Problem 1

Wang Xiwei

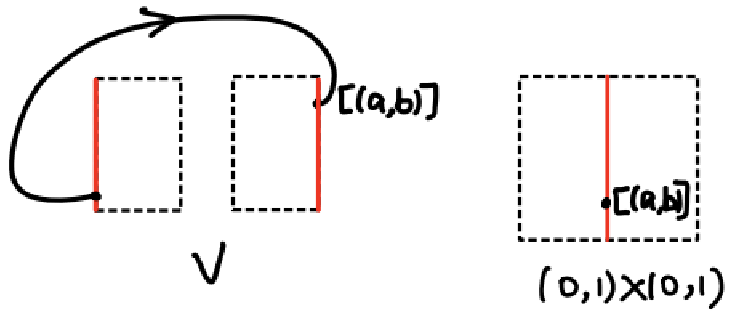
(a) Locally Euclidean: $M = [0,1] \times (0,1)/\sim$ where $(0,x) \sim (1, 1-x)$. Let $\pi\colon [0,1] \times (0,1) \to M$ be the projection map.
Define $U := \pi((0,1) \times (0,1))$ open in the quotient topology. Since $(0,1) \times (0,1)$ is saturated open in $[0,1] \times (0,1)$. Define chart $\varphi\colon U \to (0,1) \times (0,1)$ with $\varphi([x]) = x$ and $\varphi \circ \pi = \pi \circ \varphi = \text{id}$. φ is continuous by characteristic property of quotient map.
Define $V := \pi([0,1] \times (0,1) - \{1/2\} \times (0,1))$ open since the set $[0,1] \times (0,1) - \{1/2\} \times (0,1)$ saturated open in $[0,1] \times (0,1)$. Define the chart $\psi\colon V \to (0,1) \times (0,1)$ with

$$\psi([(a,b)]) = \begin{cases} (a - 1/2, b) \text{ if } a \in (1/2, 1) \\ (a + 1/2, 1 - b) \text{ if } a \in (0, 1/2) \\ (1/2, b) \text{ if } a = 0 \end{cases}$$

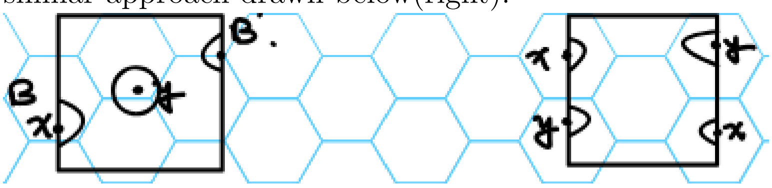
which is continuous again by characteristic property. The inverse is $\psi^{-1}\colon (0,1) \times (0,1) \to V$ with

$$\psi^{-1}((u,v)) = \begin{cases} [(u + 1/2, v)] \text{ if } u \in (0, 1/2) \\ [(u - 1/2, 1 - v)] \text{ if } u \in (1/2, 1) \\ [(0, v)] \text{ if } u = 1/2 \end{cases}$$

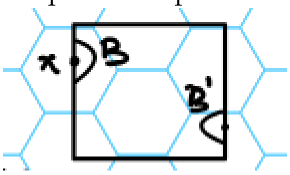
which is continuous as composition of continuous maps and quotient maps and $\varphi \circ \psi = \psi \circ \varphi = \text{id}$. Thus $(U, \varphi), (V, \psi)$ are charts covering M.
Note the first map is the operation of cutting and pasting the two pieces into one square drawn as below. And then easily find its inverse map.



Hausdorff: Assume $[x], [y] \in U$. Then $x, y \in (0,1) \times (0,1)$. There are open sets separating them and do not intersect $\{0\} \times (0,1) \cup \{1\} \times (0,1)$. They are saturated and their projection are open. If $[x] \in \pi(\{0\} \times (0,1))$ and $[y] \in U$. Then take one ball and one open half ball B to separate them. Then complement another open half ball B' to get $B \cup B'$. $B \cup B'$ is open and saturated. Choose B small enough such that $B \cup B'$ does not intersect y's open ball. Then their projection are open balls separating them, see figure below(left). If $[x], [y] \in \pi(\{0\} \times (0,1))$. Then use similar approach drawn below(right).



Second countable: This quotient map is not open. We cannot use $[0,1]\times(0,1)$ second countable to directly get M is second countable. We need to show it by definition. Consider the collection of balls of $[0,1]\times(0,1)$ with rational radii and coordinates excluding those intersecting the boundary. Now add the collection $\mathbb{W}$ to it where $\mathbb{W}$ contains the half open ball centered at the boundary complemented with the other half ball. For instance $B\cup B'$ drawn below. Now the new collection is countable many with their projections open. And their projections are a basis of quotient space.



(b) $\mathbb{T}^n=\mathbb{S}^1\times\cdots\times\mathbb{S}^1$ is the product manifold.
(c) Locally Euclidean:$\pi\colon\mathbb{R}^{n+1}-\{0\}\to\mathbb{RP}^n$. Define $\widetilde{U_i}$ to be the set with i-th coordinate $x_i\neq 0$. Then $U_i:=\pi(\widetilde{U_i})$ is open since $\widetilde{U_i}$ is saturated open. Define $\varphi_i\colon U_i\to\mathbb{R}^n$. $\varphi_i([x_1,\dots,x_{n+1}])=(x_1/x_i,\dots,x_{i-1}/x_i,x_{i+1}/x_i,\dots,x_{n+1}/x_i)$ which is continuous by characteristic property. The inverse $\psi_i(x_1,\dots,x_n)=[x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n]$ continuous as the composition of continuous maps and quotient map. $\varphi_i\circ\psi_i=\psi_i\circ\varphi_i=\mathrm{id}$. Thus (φ_i,U_i) is the chart covering $\mathbb{RP}^n$.
Second countable: The quotient map here is open. Assume open sets $U\in\mathbb{R}^{n+1}-\{0\}$. Then $\pi^{-1}\pi(U)=\bigcup_{\lambda\in\mathbb{R}-\{0\}}\lambda U$ open since $\lambda U:=\{\lambda x|x\in U\}$ open. Thus π is open. $\mathbb{R}^{n+1}-\{0\}$ is second countable, $\mathbb{RP}^n$ is second countable.
Hausdorff: Lemma: If π is open quotient map, $X/\sim$ is Hausdorff iff (x_1,x_2) is closed in $X\times X$ where $x_1\sim x_2$. In our case, $x_1\sim x_2$ iff x_1 and x_2 lie in one ray. Just to show the space where $x_1\nsim x_2$ is open which is equivalent to x_1,x_2 doe not lie in one ray. We can extend x_1,x_2 to a basis $(x_1,x_2,\dots,x_{n+1})$ which is linearly independent $\Leftrightarrow\det([x_1,\dots,x_{n+1}])\neq 0$ which is open since det continuous.
(d) The same as $\mathbb{RP}^n$
(e) $GL(n,\mathbb{R})$ is open since it is the preimage of $\det(\bullet)\neq 0$.
(f) Define the smooth map $F\colon A\mapsto A^TA$. $D_AF(X)=\frac{d}{dx}|_{t=0}(A+tX)^T(A+tX)=A^TX+X^TA$. Since $A\in GL(n,\mathbb{R})$ and $X\in M_{n\times n}(\mathbb{R})$, then A^TX can be any matrix in $M_{n\times n}(\mathbb{R})$. The image of $D_AF(X)$ is the set $\{X^T+X|X\in M_{n\times n}(\mathbb{R})\}$ which is of dimension $n(n+1)/2$ for all $A\in GL(n,\mathbb{R})$. Thus by constant rank theorem $O(n,\mathbb{R})$ is a regular submanifold with dimension $n^2-n(n+1)/2$. $SO(n,\mathbb{R})=\det^{-1}(1)\cap O(n,\mathbb{R})=\det^{-1}((0,+\infty))\cap O(n,\mathbb{R})$ is open in $O(n,\mathbb{R})$. Thus $SO(n,\mathbb{R})$ is a submanifold with dimension $n^2-n(n+1)/2=n(n-1)/2$.

PROBLEM 1A

Vladimir Nowak

Since the Möbius band M is an open subset of the Klein bottle K it suffices to show that K is a 2-manifold, since every open subset of a manifold is a manifold in its own right. We use the following proposition from covering theory:

proposition If $\overline{M}$ is a connected topological manifold, and $\Gamma < \text{Homeo}\left(\overline{M}\right)$ is a group that acts freely and properly discontinuously on $\overline{M}$, then $\overline{M}/\Gamma$ is a topological manifold (of the same dimension) and $p\colon \overline{M} \to \overline{M}/\Gamma$ is a covering map, with $\overline{M}/\Gamma = \overline{M}/\sim$ where $p \sim q$ if $\exists \gamma \in \Gamma$ such that $\gamma(p) = q$. proposition
As our manifold, we take $\overline{M} = \mathbf{R}^2$, where $p = (x, y)$ denote standard coordinates. Let $\Gamma < \text{Homeo}(\mathbf{R}^2)$ be the subgroup generated by the maps $(x, y) \mapsto (x, y+1)$ and $(x, y) \mapsto (x+1, -y)$. They certainly act freely on $\mathbf{R}^2$, as the former is a non-trivial translation and the latter the composition of a non-trivial translation along the x-axis with a reflection along the x-axis, and thus for $\gamma \in \Gamma - \text{id}$ we have $\gamma(p) \neq p$.
Since by Heine-Borel every compact set is closed and bounded on $\mathbf{R}^2$ if we show that for all $\overline{B}_R(0)$ with $R > 0$ (in the standard metric) the set $\{\gamma \in \Gamma : \gamma\left(\overline{B}_R(0)\right) \cap \overline{B}_R(0) \neq \emptyset\}$ is finite then Γ acts properly discontinuously. This is certainly the case, because if we have shifted by more than $2R$ from 0 in either the x or y direction, we don't intersect $\overline{B}_R(0)$ anymore. As $\mathbf{R}^2/\Gamma = K$ we are done.

PROBLEM 2

a). As image of a compact space, the space X is compact itself. For locally Euclidian and second countable base we take open intervals around points (red lines in the picture). The Excision Theorem with respect to complement of such an interval gives an isomorphism

$$H_1(X, X \setminus x) \cong H_1(X \setminus Z, (X \setminus Z) \setminus x) \cong H_1(S^1).$$

The complement to double 1 is connected as an open interval, any of the 1s can be connected with -1 by a path that is the image of the semicircle in the corresponding circle.
Any neighbourhoods of two 1s intersect and therefore X is not Hausdorff.

b). Consider the reduces Mayer-Vietoris sequence for A and B being images of $S^1 \times \{0\}$ and $S^1 \times \{1\}$ under the canonical projection $S^1 \times \{0, 1\} \to X$:

$$\cdots \to \widetilde{H}_1(A \cap B) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(X) \to \widetilde{H}_0(A \cap B) \to \cdots$$

As A and B are both homeomorphic to S^1 and their intersection is on open interval, we have the sequence

$$\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to 0 \to \cdots$$

with the middle arrow $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ being an isomorphism. Therefore the arrow $\mathbb{Z} \oplus \mathbb{Z} \to H_1(X)$ is an isomorphism as well.

c). The induction in Step 2 does not work, as we used the fact the intersection of two compacts is compact again. This does not necessarily hold in a non-Hausdorff space.
For X, the complement of each of the 1s is a compact space (homeomorphic to S^1), but their intersection is an open interval, thus not compact.

Problem 3

For a point $\widetilde{x} \in M$, let $D_{\widetilde{x}}$ its an open neighbourhood which maps homeomorphically onto its image along the canonical projection $p\colon M \to M/G$. For the locally consistent choice of local orientations we take classes $\alpha_x \in H^n(M/G, M/G \setminus \{x\})$ that correspond to the given orientation under isomorphisms

$$H^n(M/G, M/G\setminus\{x\}) \cong H^n(p(D_{\widetilde{x}}), p(D_{\widetilde{x}})\setminus x) \cong H^n(D_{\widetilde{x}}, D_{\widetilde{x}}\setminus\{\widetilde{x}\}) \cong H^n(M, M\setminus\{\widetilde{x}\}),$$

where $p(\widetilde{x}) = x$.

Problem 4

Vladimir Nowak
Assume M is non-orientable. This is equivalent to saying that the orientation double cover $\widetilde{M}$ is connected.
Since M is assumed to be connected, and is locally path connected (take a coordinate ball) it is path connected. Furthermore, since M is locally Euclidean, it is semi-locally simply connected. Thus M possesses a universal cover M_U.
By the classification of connected covering spaces for spaces that possess a universal cover, the equivalence classes of covering spaces of M are in one-to-one correspondence to the subgroups of $\pi_1(M)$, where for any subgroup $G < \pi_1(M)$ there exists an equivalence class of covering maps $p\colon X \to M$, s.t. $p_\#(\pi_1(X)) = G$.
The index of these subgroups corresponds to the cardinality of the fiber of the covering. Since $p\colon \widetilde{M} \to M$ is a $2:1$ covering, $\pi_1(M)$ has a subgroup of index 2.

Problem 5

We have to check the axioms for graded right-modules. Let us write x, y for elements of the total homology, and r, s for elements of the cohomology ring, and ε for the unit of the cohomology ring.

- $x \frown \varepsilon = x$. This is Proposition 8.1 (2).
- $x \frown (r \smile s) = (x \frown r) \frown s$. This follows from Proposition 8.1 (3).
- $(x + y) \frown r = x \frown r + y \frown r$. This follows from Proposition 8.1 (1).
- $x \frown (r + s) = x \frown r + x \frown s$. This follows from Proposition 8.1 (1).
- If x and r are homogeneous, then so is xr, and $\deg x \frown r = \deg x + \deg r$. This follows immediately from the definition of the cap product.

Problem 6

a). The answer is *yes*.
Indeed, let $t \in H^2(M;\mathbb{Q})$ be a generator. By Poincaré duality for M, there is a class $\alpha t \in H^2(M;\mathbb{Q})$ such that $t \cdot \alpha t = \alpha t^2 \neq 0$. In particular, $0 \neq t^2 \in H^4(M;\mathbb{Q})$. Taking t^2 as the generator of $H^4(M;\mathbb{Q})$ gives an isomorphism of graded rings

$$H^*(M;\mathbb{Q}) \cong \mathbb{Q}[t]/(t^3) \cong H^*(\mathbb{C}P^2;\mathbb{Q}).$$

b). The answer is *no.*
Consider $M = S^2 \times S^4$. There is a CW-structure on M with one cell in dimensions $0, 2, 4$ and 6 and no other cells. As all the differentials are trivial, $H^2(S^2 \times S^4; \mathbb{Q}) \cong \mathbb{Q}$. Projection of $S^2 \times S^4$ onto the first factor induces a ring homorphism $H^*(S^2; \mathbb{Q}) \to H^*(S^2 \times S^4; \mathbb{Q})$ that is an isomorphism on the second cohomology group. In particular, any element in the second cohomology of M is zero once squared and

$$H^*(S^2 \times S^4; \mathbb{Q}) \ncong \mathbb{Q}[t]/(t^4) \cong H^*(\mathbb{C}P^3; \mathbb{Q}).$$

PROBLEM 7

Let $f\colon \mathbb{C}P^3 \to \mathbb{C}P^3$ be a continuous map of degree d. For $f^*\colon H^2(\mathbb{C}P^3) \to H^2(\mathbb{C}P^3)$ define d' by $f^*(x) = d'x$, where $x \in H^2(\mathbb{C}P^3)$ is a generator. From our computation of $H^\bullet(\mathbb{C}P^3)$, we know that $x \cup x \cup x$ is a generator of $H^6(\mathbb{C}P^3)$, which evaluates to 1 on the fundamental class $[\mathbb{C}P^3]$. We have

$$\begin{aligned}
(d')^3 &= \mathrm{ev}(d'x \smile d'x \smile d'x, [\mathbb{C}P^3]) \\
&= \mathrm{ev}(f^*(x \smile x \smile x), [\mathbb{C}P^3]) \\
&= \mathrm{ev}(x \smile x \smile x, f_*([\mathbb{C}P^3])) \\
&= \mathrm{ev}(x \smile x \smile x, d[\mathbb{C}P^3]) \\
&= d.
\end{aligned}$$

Therefore, the degree has to be a cube, in particular, the answer to **a)** is *no.*
For **b)**, consider the map $f\colon \mathbb{C}P^3 \to \mathbb{C}P^3$ given by the formula:

$$f\big([z_0 : \cdots : z_3]\big) \coloneqq [z_0^2 : \cdots : z_3^2].$$

Applying **Problem 9a)** to the point $x = [1 : 1 : 1 : 1]$ we get that the degree of f is exactly 8.
In (real) local coordinates in chart U_3 the map is given by

$$(x_0, y_0, x_1, y_1, x_2, y_2) \mapsto (x_0^2 - y_0^2, 2x_0y_0, x_1^2 - y_1^2, 2x_1y_1, x_2^2 - y_2^2, 2x_2y_2),$$

which has degree one in the given preimages.

PROBLEM 8

no solutions for starred problems

Problem 9

Jeyakumar Aparna

(a) Let $y \in B$ and $x_i \in B_i$ be such that $f(x_i) = y$. We fix the following notation,

$$\begin{array}{ccccc} H_n(M) & \longrightarrow & H_n(M, M \setminus x_i) & \xleftarrow{\cong} & H_n(B_i, B_i \setminus x_i) \\ [M] & \longmapsto & \mu_{x_i} & \longmapsto & \sigma_{x_i} \\ H_n(N) & \longrightarrow & H_n(N, N \setminus y) & \xleftarrow{\cong} & H_n(B, B \setminus y) \\ [N] & \longmapsto & \mu_y & \longmapsto & \sigma_y \end{array}$$

and $f_*([M]) = \deg f.[N]$, $f_*(\sigma_{x_i}) = \epsilon_i \sigma_y$. Consider the following commutative diagram.

$$\begin{array}{ccccc} & & H_n(B_i, B_i \setminus x_i) & \xrightarrow{f_*} & H_n(B, B \setminus y) \\ & \swarrow^{\cong} & \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \cong} \\ H_n(M, M \setminus x_i) & \xleftarrow{\mathrm{incl}_*} & H_n(M, M \setminus \{x_i\}_i) & \xrightarrow{f_*} & H_n(N, N \setminus y) \\ & \nwarrow_{\mathrm{incl}_*} & \uparrow{\scriptstyle \mathrm{incl}_*} & & \uparrow{\scriptstyle \mathrm{incl}_*} \\ & & H_n(M) & \xrightarrow{f_*} & H_n(N) \end{array}$$

The two isomorphisms are due to excision. Again by excision, we see that

$$H_n(M, M \setminus \{x_i\}_i) \cong \bigoplus_i H_n(B_i, B_i \setminus x_i)$$

Then, we have the following mappings.

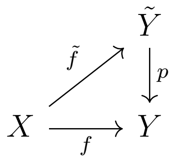
$$\begin{array}{ccccc} & & \sigma_{x_i} & \xmapsto{f_*} & \epsilon_i \sigma_y \\ & \swarrow & & & \\ \mu_{x_i} & \longleftarrow & \sum_j \mu_{x_j} & \xmapsto{f_*} & \sum_i \epsilon_i \mu_y \\ & \nwarrow & \uparrow & & \uparrow \\ & & [M] & \xmapsto{f_*} & \deg f.[N] \end{array}$$

Due to the commutativity of the lower square, $\deg f = \sum_i \epsilon_i$.

(b) Suppose $f : X \to Y$ is a p-sheeted covering map. Then, there is an open cover $\{V_i\}_i$ of Y such that $f^{-1}(V_i) = \sqcup_{j=1}^{p} U_j^i$ and f restricted to each U_j^i is a homeomorphism onto V_i.
Define a map $\phi : X \to \{\pm 1\}$ as follows: For $x \in X$, choose V_i with $f(x) \in V_i$ and

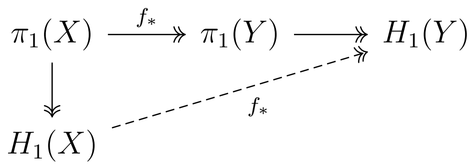
$x \in U_k^i$ if $f^{-1}(V_i) = \sqcup_{j=1}^p U_j^i$ and set $\phi(x) = \epsilon_x$ where $\epsilon_x = \pm 1$ according to whether $f : U_k^i \to V_i$ is orientation preserving or reversing. The definition does not depend on the choice of the open set V_i.
Note that ϕ is locally constant. Since M is connected, ϕ is constant. Let $y \in Y$ and $f^{-1}(y) = \{x_1, ..., x_p\}$. From (a), deg $f = \sum_{j=1}^p \epsilon_{x_j} = \sum_{j=1}^p \epsilon_{x_1} = \pm p$.

(c) (We follow the hint given in Hatcher, page 258):
Let $\tilde{Y}$ be a covering space over Y with fundamental group $\text{Im}(f_*) \subset \pi_1(Y)$. Then, there exists a lift of f, that is, $\tilde{f} : X \to \tilde{Y}$ such that the following diagram



commutes. Now, $f = p \circ \tilde{f}$ implies that deg$f = 1 =$ deg p deg $\tilde{f}$. So, deg $p = \pm 1$. If the covering $\tilde{Y}$ is finite-sheeted, then (b) gives that $\tilde{Y}$ is a one-sheeted covering of Y. Hence, Imf_* is an index 1 subgroup of $\pi_1(Y)$, that is, f_* is surjective. If $\tilde{Y}$ is $\infty-$sheeted covering then $\tilde{Y}$ is non-compact which implies that $H_n(\tilde{Y}) = 0$. This contradicts the fact that deg $p = \pm 1$.

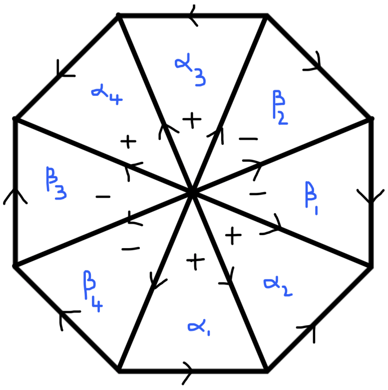
(d) Firstly, we claim that in the situation as above, we also have that $f_* : H_1(X) \to H_1(Y)$ is also surjective. Indeed, we have the following commutative diagram

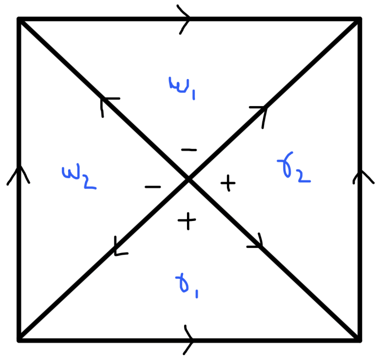


where the dotted line is due to the universal property associated with abelianization. From the commutativity of the diagram, we get that $f_* : H_1(X) \to H_1(Y)$ is surjective.

($\Longrightarrow$) Suppose there is a continuous map $f : \Sigma_g \to \Sigma_h$ of degree 1. Then, using (c), $f_* : H_1(\Sigma_g) \to H_1(\Sigma_h)$ is surjective. That is, there is a surjective map $\phi : \mathbb{Z}^{2g} \to \mathbb{Z}^{2h}$. Then, $g \geq h$.
One way to see this is by tensoring the sequence $\mathbb{Z}^{2g} \to \mathbb{Z}^{2h} \to 0$ by a field over $\mathbb{Z}$, say $\mathbb{Q}$. Then, $\mathbb{Z}^{2g} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Z}^{2h} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$ is exact and we get a surjective map $\mathbb{Q}^{2g} \to \mathbb{Q}^{2h}$. Now, using dimension argument for vector spaces over $\mathbb{Q}$, we get the inequality.

($\Longleftarrow$) Suppose $g \geq h$. We have the $\Delta-$ complex structure on Σ_g as shown in the picture. Then, $[\Sigma_g]$ is represented by the sum of all $4g$ 2-simplices with the indicated sign.





Define $f : \Sigma_g \to \Sigma_h$ on each of the 2-simplex as follows:
$\alpha_i \mapsto \gamma_i$ $\beta_i \mapsto \omega_i$ for $1 \leq i \leq 2h$
$\alpha_{2h+i} \mapsto \gamma_1$ $\beta_{2h+i} \mapsto \omega_2$ for $1 \leq i \leq 2g - 2h - 1$ and i odd
$\alpha_{2h+i} \mapsto -\omega_2$ $\beta_{2h+i} \mapsto -\gamma_1$ for $1 \leq i \leq 2g - 2h$ and i even.
Then,

$$\begin{aligned} f_*([\Sigma_g]) &= f_*(\sum_{i=1}^{2g}([\alpha_i] - [\beta_i])) \\ &= \sum_{i=1}^{2g}[f(\alpha_i)] - \sum_{i=1}^{2g}[f(\beta_i)] \\ &= \sum_{i=1}^{2h}\gamma_i - \sum_{i=1}^{2h}\omega_i \\ &= [\Sigma_h] \end{aligned}$$

Hence, f is a continuous map with degree 1.