Problem 1

By definition,

$$\varinjlim \left(\cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \right) = \bigoplus_i \mathbb{Z} / \langle n_i = 2n_{i-1} \rangle$$

Let $\mathbb{Z}\left[\frac{1}{2}\right] \subseteq \mathbb{Q}$ be a subring of fractions of the form

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \right\}$$

Then we have that $\varinjlim \left(\cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \right)$ is isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$ with the morphisms

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{\times 2} & \cdots \\ & \searrow^{1} & \downarrow^{\frac{1}{2}} & \swarrow_{\frac{1}{4}} & & \swarrow & \\ & & \mathbb{Z}\left[\frac{1}{2}\right] & & & & \end{array}$$

Problem 2

Wang Xiwei

$C_i(U_\alpha) = \{c_i | \text{im } c_i \subset U_\alpha\} \subset C_i(X)$. So we have an inclusion

$$j : C_i(U_\alpha) \to C_i(X)$$

By the universal property, there exists a unique homomorphism $f : \varinjlim H_i(U_\alpha; G) \to H_i(X; G)$ with

$$\begin{array}{ccc} H_i(U_\alpha; G) & & \\ \downarrow^{g_\alpha} & \searrow^{j_*} & \\ \varinjlim H_i(U_\alpha; G) & \xrightarrow{f} & H_i(X; G) \end{array}$$

f is surjective: Assume $[c] \in H_i(X; G)$, that is im $c \subset X$. The image of a chain as a finite sum of simplex is compact, it is contained in some U_α. Thus $[c] \in H_i(U_\alpha; G)$ and $j_*([c]) = [j(c)] = [c]$. $f(g_\alpha([c])) = j_*([c]) = [c]$.

f is injective: Assume $x \in \varinjlim H_i(U_\alpha; G)$ with $f(x) = 0$. By construction of direct limit there is some U_α such that $x = g_\alpha([\varphi])$ with $[\varphi] \in H_i(U_\alpha; G)$. Then $[j(\varphi)] = j_*([\varphi]) = f(x) = 0$.(Note: we cannot claim $[\varphi] = 0$ here since j_* is not necessarily injective). There exists $\psi \in C_{i+1}(X)$ such that $d\psi = j(\varphi)$. Pick some U_β containing the image of ψ, $\psi \in C_{i+1}(U_\alpha \cup U_\beta)$. We have the following commutative diagram,

$$\begin{array}{ccc} \psi \in C_{i+1}(U_\alpha \cup U_\beta) & \xrightarrow{j} & \psi \in C_{i+1}(X) \\ \downarrow^{d} & & \downarrow^{d} \\ \varphi \in C_i(U_\alpha \cup U_\beta) & \xrightarrow{j} & \varphi \in C_i(X) \end{array}$$

Thus $[\varphi] = [d\psi] = [dj(\psi)] = [jd(\psi)] = j_*[d(\psi)] = j_*[\varphi] = 0 \in H_i(U_\alpha \cup U_\beta)$. $x = g_\alpha([\varphi]) = 0$.

PROBLEM 3

Vladimir Nowak
Proof by contradiction. Suppose $H_c^0(X;G) \neq 0$. Then $\exists \varphi \in C_c^0(X;G) - 0$ such that $d^0\varphi = 0$. Since φ is a compactly supported cochain, per construction there exists a $K \subset X$ compact such that

$$\varphi(\sigma) = 0, \forall \sigma\colon \triangle^0 \to X \text{ s.t. } \operatorname{im}(\sigma) \cap K = \emptyset$$

Since X is non-compact, $X - K$ is non-empty, i.e. $\exists x \in X - K$. Furthermore, as φ is non-trivial, $\exists y \in K$ such that $\varphi(y) \neq 0$. Let $\tau\colon \triangle^1 \to X$ be a path from x to y (which exists due to the assumption of path-connectedness). Then we have $d^0\varphi(\tau) = \varphi(d_1(\tau)) = \varphi(y) \neq 0$, in contradiction to $d^0\varphi = 0$.

PROBLEM 4

a). By Poincaré duality, we have

$$H_c^k(\mathbb{R}^n \setminus \{0\};\mathbb{Z}) \cong H_{n-k}(\mathbb{R}^n \setminus \{0\};\mathbb{Z}) \cong H_{n-k}(S^{n-1};\mathbb{Z}),$$

where the last isomorphism holds as $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

b). *no solutions for starred problems*

PROBLEM 5

Vladimir Nowak
Note that $\varphi(S^n) \subset S^{n+1}$ is compact as the image of a compact set under a continuous map. Furthermore, as S^n and S^{n+1} are both compact, any continuous map between them is proper. Due to our assumption on φ being injective, φ is an injective proper map, which is equivalent to φ being a closed embedding, where we use that S^{n+1} is Hausdorff and locally compact as a topological manifold. Since S^n is locally Euclidean, it is locally contractible, and since $\varphi(S^n)$ is homeomorphic to S^n it is also locally contractible. Since $\varphi(S^n) \neq \emptyset$, S^{n+1} due to dimensionality reasons of the given manifolds we can apply Alexander duality, which gives:

$$\widetilde{H}_0\left(S^{n+1} - \varphi(S^n)\right) \cong \widetilde{H}^{n-1}(\varphi(S^n)) \cong \mathbf{Z}.$$

Due to the definition of reduced homology, we get that $S^{n+1} - \varphi(S^n)$ has two connected components.