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Alg Top I      Top. Space  $X$   
 $\Downarrow$

Singular Chain Complex  $C(X) = \dots \rightarrow C_n(X) \xrightarrow{d_n} C_{n-1}(X) \rightarrow \dots \rightarrow C_0(X) \rightarrow 0$   
 $\Downarrow$

Homology groups  $H_i(X)$

Alg Top II      Spice up  $C(X)$  before taking homology  
 to get more sensitive invariants and more geom. applications

Topics: \* Homology with Coefficients (for abelian groups  $M$  define chain complex  $C(X) \otimes M$

with homology groups  $H_i(X; M)$ )

\* Cohomology (cochain complex  $\text{Hom}(C(X), M)$  with cohomology groups  $H^i(X; M)$ )

\* Poincaré Duality for compact  $n$ -dim manifolds  $X$

( $H_i(X; M) \cong H^{n-i}(X; M)$ , leading to intersection forms  $H_{n/2}(X) \times H_{n/2}(X) \rightarrow \mathbb{Z}$  for even  $n$ )

Color Scheme: Sections, Date

Def / Thm / Proof etc.

Newly defined terms

References

Corrections

# ① Tensor Products of modules (Spanier: Intro, Sec 4 & 5)

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Ch 5 Sec 1; Hatcher Sec 3.2 / A Künneth formula; Atiyah-MacDonald  
Ch 2 / Tensor Product of modules)

Let  $R$  be a commutative ring with 1 (after this section only  $R = \mathbb{Z}$ ).

**Prop 1** Let  $M, N$  be  $R$ -modules. Then there exists an  $R$ -module  $T$  and a bilinear map  $\mu: M \times N \rightarrow T$  such that:

For all  $R$ -modules  $K$  and bilinear maps  $f: M \times N \rightarrow K$  there is a unique homomorphism  $g: T \rightarrow K$  with  $g \circ \mu = f$ .

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & K \\ \mu \downarrow & \searrow \exists! g & \uparrow \\ T & \xrightarrow{\quad} & K \end{array}$$

**Proof**  $U :=$  free  $R$ -module with basis the set  $M \times N$ .

$I :=$  submodule of  $U$  generated by

$$\left\{ (\lambda x + x', y) - \lambda(x, y) - (x', y) \mid \lambda \in R, x, x' \in M, y \in N \right\} \\ \cup \left\{ (x, \lambda y + y') - \lambda(x, y) - (x, y') \mid \lambda \in R, x \in M, y, y' \in N \right\}$$

Let  $T = U/I$  and  $\mu: M \times N \rightarrow T$ ,  $\mu(x, y) = [(x, y)]$

Check that  $\mu$  is bilinear! Now let  $f: M \times N \rightarrow K$  as above be given.

Check existence of  $g$ :

Let  $\tilde{g}: U \rightarrow K$  be the homomorphism with  $\tilde{g}((x, y)) = f(x, y)$ .

Check that  $I \subseteq \ker \tilde{g} \Rightarrow \tilde{g}$  induces  $g: T \rightarrow K$ .

We have  $g(\mu(x, y)) = g([(x, y)]) = \tilde{g}((x, y)) = f(x, y) \checkmark$

Check uniqueness of  $g$ :

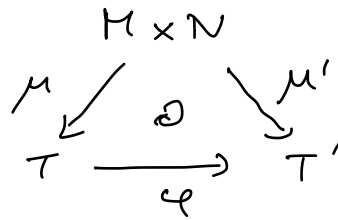
If  $g': T \rightarrow K$  with  $g' \circ \mu = f$ , then  $g'([(x, y)]) = g([(x, y)])$

for all  $x \in M, y \in N$ . But such  $[(x, y)]$  generate  $T \Rightarrow g = g' \checkmark$

□



**Prop 2** If  $\mu: M \times N \rightarrow T$  and  $\mu': M \times N \rightarrow T'$  both satisfy the condition in Prop 1, then there is a unique isomorphism  $\varphi: T \rightarrow T'$  such that  $\varphi \circ \mu = \mu'$ .



**Proof** By assumption (existence of  $g$ ),  $\exists \varphi: T \rightarrow T'$  with  $\varphi \circ \mu = \mu'$  and  $\exists \psi: T' \rightarrow T$  with  $\psi \circ \mu' = \mu$ . Then  $\psi \circ \varphi: T \rightarrow T$  with  $\psi \circ \varphi \circ \mu = \mu$ . By assumption (uniqueness of  $g$ )  $\Rightarrow \psi \circ \varphi = \text{id}_T$ .

Similarly  $\varphi \circ \psi = \text{id}_{T'}$ . □

**Def**  $T$  as in Prop 1 is called the **tensor product of  $M$  and  $N$  over  $R$** , written  **$M \otimes_R N$** . Drop  $R$  if there is no ambiguity. Write  **$x \otimes y = \mu(x, y) \in M \otimes_R N$** .

**Notation**  $\times$  and  $\oplus$  is the same for finitely many modules.

- Prop 3**
- (1)  $\exists$  iso  $M \otimes N \rightarrow N \otimes M$  with  $x \otimes y \mapsto y \otimes x$ .
  - (2)  $\exists$  iso  $(M \oplus N) \otimes K \rightarrow (M \otimes K) \oplus (N \otimes K)$  with  $(x, y) \otimes z \mapsto (x \otimes z) + (y \otimes z)$
  - (3)  $I \subseteq R$  ideal  $\Rightarrow \exists$  iso  $(R/I) \otimes M \rightarrow M/IM$  with  $r \otimes m \mapsto [rm]$

**Remark 4** Special case of (3): iso  $R \otimes M \rightarrow M$ ,  $r \otimes m \mapsto rm$ .

# Proof of Prop 3 (1)

$$\begin{array}{ccc} M \oplus N & \xrightarrow{h: (x,y) \mapsto (y,x)} & N \oplus M \\ \mu_{M,N} \downarrow & \uparrow & \downarrow \mu_{N,M} \\ M \otimes N & \xrightarrow{\exists! \varphi} & N \otimes M \end{array}$$

Let  $h: M \oplus N \rightarrow N \oplus M$  be the homom. with  $(x,y) \mapsto (y,x)$ .

Then  $\mu_{N,M} \circ h: M \oplus N \rightarrow N \otimes M$  is bilinear.

By the universal property of  $\otimes$ ,  $\exists \varphi: M \otimes N \rightarrow N \otimes M$  with  $\varphi \circ \mu_{M,N} = \mu_{N,M} \circ h$ , ie  $\varphi(x \otimes y) = y \otimes x$ .

Let  $\psi$  be the analogous homo with  $M, N$  switched  $\Rightarrow$

$\varphi, \psi$  are mutually inverse homomorphisms.

In the lecture, a similar (but incorrect) proof was given, based on the erroneous assumption that  $h$  is bilinear (it is, in fact, linear).

Proof of (2)-(4): Exercises. □

**Remark 5** Using Prop 3, we can calculate  $M \otimes_{\mathbb{Z}} N$  for all finitely generated abelian groups  $M, N$ .

**Example 6**  $\mathbb{Z}^2 \otimes \mathbb{Z}^2 = (\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}^4$

So  $\mathbb{Z}^2 \otimes \mathbb{Z}^2$  is free with basis  $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$ .

Careful! Not every element of  $\mathbb{Z}^2 \otimes \mathbb{Z}^2$  is of the form  $x \otimes y$ ,

eg  $e_1 \otimes e_1 + e_2 \otimes e_2$  isn't (and isn't equal to  $(e_1 + e_2) \otimes (e_1 + e_2)$ ).

**Remark 7** (1) Every element of  $M \otimes N$  is equal to  $\sum_{i=1}^n x_i \otimes y_i$  for some finite  $n$ ,  $x_i \in M$ ,  $y_i \in N$ .

(2)  $(\lambda x) \otimes y = x \otimes (\lambda y)$

(3)  $(x + x') \otimes y = x \otimes y + x' \otimes y$

Prop 8  $f: M \rightarrow N$ ,  $f': M' \rightarrow N'$   $R$ -module homomorphisms.

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(1)  $\exists$  homo  $f \otimes f': M \otimes M' \rightarrow N \otimes N'$  with  $x \otimes x' \mapsto f(x) \otimes f'(x')$ .

(2)  $(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$ .

(3)  $(f + g) \otimes f' = f \otimes f' + g \otimes f'$  and similarly in second factor.

Proof (1) Induced by the bilinear map  $M \times M' \rightarrow N \otimes N'$ ,  
 $(x, x') \mapsto f(x) \otimes f'(x')$ .

(2), (3) Check that  $x \otimes x'$  has the same image under both maps.  $\square$

Prop 9  $M$  an abelian group,  $S$  a commutative ring. Then  $M \otimes_{\mathbb{Z}} S$

carries an  $S$ -module structure given by  $s \cdot (x \otimes t) = x \otimes st$ .

For homom  $f: M \rightarrow N$  and  $S$ -homom  $g: S \rightarrow S$ ,

$f \otimes g: M \otimes S \rightarrow N \otimes S$  is an  $S$ -homom.

Proof: Exercise (careful: why is the function  $x \otimes t \mapsto x \otimes st$  well-def?).

Category theory intermezzo

Weibel Sec 1.1, 1.2

Reminder A Category  $\mathcal{C}$  consists of a class  $|\mathcal{C}|$  of objects, for all  $X, Y \in |\mathcal{C}|$  a set  $\mathcal{C}(X, Y)$  of morphisms with a distinguished identity morphism  $1_X \in \mathcal{C}(X, X)$ , and

composition functions  $\circ: \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$

such that  $(f \circ g) \circ h = f \circ (g \circ h)$  and  $f \circ 1_X = 1_Y \circ f = f$ .

A (covariant) functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of functions

$|\mathcal{C}| \rightarrow |\mathcal{D}|$  and  $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  with

$F(f \circ g) = Ff \circ Fg$  and  $F1_X = 1_{FX}$ . For a contravariant

functor, one has instead  $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(FY, FX)$  and

$F(f \circ g) = Fg \circ Ff$ .

**Def** A **preadditive category**  $\mathcal{C}$  is a category with abelian group structures on  $\mathcal{C}(X, Y)$ , such that compositions are bilinear. A functor  $F$  between preadditive  $\mathcal{C} \rightarrow \mathcal{D}$  is **additive** if the functions  $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  (or  $\rightarrow \mathcal{D}(FY, FX)$  if  $F$  is contravariant) are linear.

**Examples 10**  $R$  commutative ring with 1.

(1) The category  $R\text{-Mod}$  of  $R$ -modules and  $R$ -homomorphisms is preadditive.

(2) Chain complex over a preadditive category  $\mathcal{C}$ :

sequence of  $C_0, C_1, \dots \in |\mathcal{C}|$  and morphisms

$d_1: C_1 \rightarrow C_0, d_2: C_2 \rightarrow C_1, \dots$  with  $d_i \circ d_{i+1} = 0$ .

The cat.  $Ch(\mathcal{C})$  of  $\mathcal{C}$ -chain complexes and chain maps is again preadditive. Chain maps  $f: C \rightarrow C'$  are sequences  $f_0, f_1, \dots$  with  $f_i \in \mathcal{C}(C_i, C'_i)$  and  $f_i \circ d_{i+1} = d'_{i+1} \circ f_{i+1}$  for all  $i \geq 0$ .

(3) Cat of Top spaces  $\rightarrow Ch(\mathbb{Z}\text{-Mod})$ ,

$X \mapsto C(X), f \mapsto f_c$  is a functor ( $Alg_{Top I}$ )

(3') Refinement of (3): Functor

Cat of Pairs of Top spaces  $\rightarrow Ch(\mathbb{Z}\text{-Mod})$

$(X, A) \mapsto C(X, A),$

$f \mapsto f_c$

Objects:  $(X, A)$  with  $X$  Top space,  $A \subseteq X$ .

Morphisms  $f: (X, A) \rightarrow (Y, B): f: X \rightarrow Y$  cont. with  $f(A) \subseteq B$ .

(4)  $C_*(R\text{-Mod}) \longrightarrow R\text{-Mod}$ ,  $C \mapsto H_i(C) := \ker d_i / \operatorname{im} d_{i+1}$   
 $f \mapsto f_*$  are additive functors for each fixed  $i \geq 0$ .

(5) Composing (3') and (4) gives functors

Pairs of top spaces  $\longrightarrow \mathbb{Z}\text{-Mod}$ ,  $(X, A) \mapsto H_i(X, A)$ ,  
 $f \mapsto f_*$ .

(6)  $M$  a fixed  $R$ -module. Then  $R\text{-Mod} \longrightarrow R\text{-Mod}$ ,

$N \mapsto N \otimes_R M$ ,  $f \mapsto f \otimes \operatorname{id}_M$  also written as  $f \otimes M$   
 is an additive functor! (see Prop 8)

(6')  $\mathbb{Z}\text{-Mod} \longrightarrow S\text{-Mod}$ ,

$M \mapsto M \otimes_{\mathbb{Z}} S$ ,  $f \mapsto f \otimes \operatorname{id}_S$

is another additive functor (see Prop 9)

② Homology with coefficients Spanier 5.1, Hatcher 2.2

$X$  top. space,  $A \subseteq X$ ,  $M$  an abelian group.

Prop 1  $\dots \xrightarrow{d_2 \otimes \operatorname{id}_M} C_1(X, A) \otimes M \xrightarrow{d_1 \otimes \operatorname{id}_M} C_0(X, A) \otimes M \longrightarrow 0$

is a chain complex.

Proof postponed.

Def We call the complex in Prop 1 the chain complex of  $(X, A)$

with coefficients in  $M$ , denoted by  $C(X, A) \otimes M$ . We call

$H_i(C(X, A) \otimes M)$  the  $i$ -th homology group with coefficients in  $M$ ,  
 denoted by  $H(X, A; M)$ .

Rmk 2  $C(X, A) \otimes \mathbb{Z}$  is naturally isomorphic to  $C(X, A)$ .

**Goal** Chain complexes & homology groups with any coefficients  $M$  have all the good properties proven for  $\mathbb{Z}$  coefficients in Alg Top I.

**Remark 4** Recall  $C_i(X)$  is a free  $\mathbb{Z}$ -module with basis the singular simplices  $\sigma: \Delta^i \rightarrow X \Rightarrow C_i(X) \otimes M \cong \bigoplus_{\sigma: \Delta^i \rightarrow X} M$ . So one may

think of a chain in  $C_i(X) \otimes M$  as a finite linear combination

with coefficients  $m_j \in M$  of singular simplices  $\sigma_i: \sum_{j=1}^k \sigma_j \otimes m_j$ .

**Def (Eilenberg-Steenrod Axioms)**, from Alg Top I

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A **homology theory** is the following.

Data: For all  $n \in \mathbb{Z}$ :

\* Functors  $h_n$  from Cat of pairs of spaces  $\rightarrow \mathbb{Z}$ -Mod.

\* Natural Homomorphisms  $\partial: h_{n+1}(X, A) \rightarrow h_n(A) := h_n(A, \emptyset)$

$$\begin{array}{ccc} h_{n+1}(X, A) & \xrightarrow{\partial} & h_n(A) \\ f_* \downarrow & & \downarrow f_* \\ h_{n+1}(Y, B) & \xrightarrow{\partial} & h_n(B) \end{array} \quad \begin{array}{l} \text{commutes for all} \\ \text{cont. } f: (X, A) \rightarrow (Y, B) \end{array}$$

Axioms: (1)  $f \simeq g \Rightarrow f_* = g_*$  (Homotopy)

(2)  $\bar{u} \subseteq A^0$ , inclusion  $i: (X \setminus u, A \setminus u) \rightarrow (X, A) \Rightarrow i_*$  iso (Excision)

(3)  $h_n(\text{one point space}) = 0$  for  $n \neq 0$  (Dimension)

(4) For inclusions  $i: X_\alpha \rightarrow \coprod_\alpha X_\alpha$ ,

$\bigoplus h_n(X_\alpha) \xrightarrow{\sum i_{\alpha*}} h_n(\coprod_\alpha X_\alpha)$  is an iso. (Additivity)

(5) There are long exact sequences (Exactness)

$$\dots \rightarrow h_n(A) \xrightarrow{\text{incl}_*} h_n(X) \xrightarrow{\text{incl}_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \rightarrow \dots$$

A more precise Goal Thm 5  $H_n(\cdot; M)$  is a homology theory.

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Prop 6  $F: \mathcal{Z}\text{-Mod} \rightarrow \mathcal{E}$  an additive functor.

(1) An additive functor  $Ch(\mathcal{Z}\text{-Mod}) \rightarrow Ch(\mathcal{E})$ , which we also denote by  $F$ , is given by sending a chain complex  $C$ .

$$F(C) = \dots \rightarrow FC_2 \xrightarrow{Fd_2} FC_1 \xrightarrow{Fd_1} FC_0 \rightarrow 0$$

and a chain map  $f: C \rightarrow C'$  to  $F(f)$  with  $F(f)_i = F(f_i)$ .

(2) If  $f, g: C \rightarrow C'$  are homotopic, then so are  $F(f)$  and  $F(g)$ .

(3)  $f: C \rightarrow C'$  a homotopy equivalence  $\Rightarrow$  so is  $Ff$ .

Proof (1)  $Fd_1 \circ Fd_2 = F(d_1 \circ d_2) = F0 = 0 \checkmark$

$$\begin{array}{ccccc} C_i & \xrightarrow{d_i} & C_{i-1} & & FC_i & \xrightarrow{Fd_i} & FC_{i-1} \\ f_i \downarrow & \hookrightarrow & \downarrow f_{i-1} & \xrightarrow{F} & Ff_i \downarrow & \hookrightarrow & \downarrow Ff_{i-1} \checkmark \\ C'_i & \xrightarrow{d'_i} & C'_{i-1} & & FC'_i & \xrightarrow{Fd'_i} & FC'_{i-1} \end{array}$$

Check that  $F$  is an additive functor.

(2)  $f \simeq g \Rightarrow \exists$  homotopy  $h: C \rightarrow C'$ , ie  $h_i: C_i \rightarrow C'_{i+1}$ ,

$$\begin{array}{ccccc} C_{i+1} & \xrightarrow{d} & C_i & \xrightarrow{d} & C_{i-1} \\ f \downarrow \parallel g & \nearrow h & f \downarrow \parallel g & \nearrow h & f \downarrow \parallel g \\ C'_{i+1} & \xrightarrow{d'} & C'_i & \xrightarrow{d'} & C'_{i-1} \end{array} \quad \text{with} \quad h d + d' h = f - g$$

$\Rightarrow Fh: FC \rightarrow FC'$  homotopy and  $Fh Fd + Fd' Fh = Ff - Fg$ .

(3)  $g: C' \rightarrow C$  and  $f \circ g \simeq \text{id}_{C'}$ ,  $g \circ f \simeq \text{id}_C \Rightarrow$

$$F(f) \circ F(g) \simeq \text{id}_{F(C')} \quad , \quad F(g) \circ F(f) \simeq \text{id}_{F(C)}$$

□

**Corollary 7** (apply Prop 6 to  $F = -\otimes M$ )

- (1)  $C(X, A) \otimes \mathbb{Z}$  is a chain complex (that was Prop 1)

- (2) Cont.  $f: (X, A) \rightarrow (Y, B)$  induce chain maps

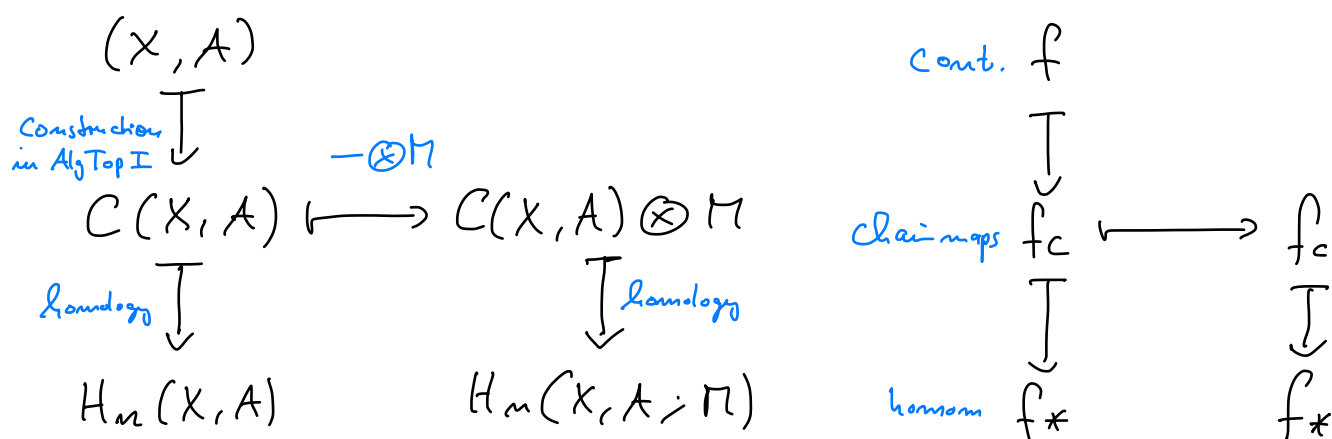
$$f_c \otimes \text{id}_M : C(X, A) \otimes M \rightarrow C(Y, B) \otimes M.$$

- $$(3) \quad f \simeq g \Rightarrow f_* \otimes M \simeq g_* \otimes M.$$

- (4)  $f_c \otimes M$  induces  $f_* : H_n(X, A; M) \rightarrow H_n(Y, B; M)$

**Notation** We'll write  $f_c$  for  $f_c \otimes \text{id}_M$ .

## Overview of functors



**Prmk 8** For a commutative ring  $S$ ,  $C(X, A) \otimes S$  is a chain complex over  $S$ ,  $H_i(X, A; S)$  is an  $S$ -module, and  $f_c$  and  $f_*$  are  $S$ -linear. Particularly useful for  $S$  a field!



We have constructed half of the data to show  $H_n(-; \mathbb{N})$  is a homology theory, and we have proved axiom (1) (Homotopy)

**Proof of Axiom (2) (Excision)**  $i_c: C(X \setminus U, A \setminus U) \rightarrow C(X, A)$

is a homotopy equivalence (Alg Top I).

$- \otimes \mathbb{N}: C(\mathbb{Z}\text{-Mod}) \rightarrow C(\mathbb{Z}\text{-Mod})$  preserves homotopy equiv. (by Prop 5(3)).

$\Rightarrow i_c \otimes \mathbb{N}$  is a hom. equiv.

$\Rightarrow i_*: H_n(X \setminus U, A \setminus U; \mathbb{N}) \rightarrow H_n(X, A; \mathbb{N})$  is an iso.

□

**Proof of Axiom (3) (Dimension)**

For  $X$  the one-point space,

$$C(X) \cong \dots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

$$\Rightarrow C(X) \otimes \mathbb{N} \cong \dots \xrightarrow{\text{id}_{\mathbb{N}}} \mathbb{N} \xrightarrow{0} \mathbb{N} \xrightarrow{\text{id}_{\mathbb{N}}} \mathbb{N} \xrightarrow{0} \mathbb{N} \rightarrow 0$$

$$\Rightarrow H_n(X; \mathbb{N}) \cong \begin{cases} \mathbb{N} & n=0 \\ 0 & \text{else} \end{cases}$$

□

**Proof of Axiom (4) (Additivity)**

$$\bigoplus_{\alpha} C(X_{\alpha}) \xrightarrow{\sum (i_{\alpha})_c} C(X) \text{ is}$$

a homotopy equiv. (Alg Top I)  $\Rightarrow$  so is  $(\bigoplus C(X_{\alpha})) \otimes \mathbb{N} \xrightarrow{(\sum (i_{\alpha})_c) \otimes \text{id}_{\mathbb{N}}} C(X) \otimes \mathbb{N}$ ,

$$\text{which is isomorphic to } \bigoplus (C(X_{\alpha}) \otimes \mathbb{N}) \xrightarrow{\sum (i_{\alpha})_c \otimes \text{id}_{\mathbb{N}}} C(X) \otimes \mathbb{N} \quad \square$$

## Construction of connecting maps $\partial$ and Proof of Axiom (5) (Exactness)

$0 \rightarrow C(A) \xrightarrow{\text{incl}_A} C(X) \xrightarrow{\text{incl}_C} C(X, A) \rightarrow 0$  is a SES of chain complexes of free abelian groups  $\Rightarrow$

$0 \rightarrow C(A) \otimes M \xrightarrow{\text{incl}_A} C(X) \otimes M \xrightarrow{\text{incl}_C} C(X, A) \otimes M \rightarrow 0$  is also exact! (Exercise)

This concludes the proof, using:

**Lemma 8** (Alg Top I) If  $0 \rightarrow C \xrightarrow{f} D \xrightarrow{g} E \rightarrow 0$  is a SES

of chain complexes over a ring, then there is a LES in homology:

$$\dots \rightarrow H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \rightarrow \dots$$

Moreover, the  $\partial$  may be chosen naturally, which means:

$$\text{If } \begin{array}{ccccccc} 0 & \rightarrow & C & \xrightarrow{f} & D & \xrightarrow{g} & E \rightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \rightarrow & C' & \rightarrow & D' & \rightarrow & E' \rightarrow 0 \end{array} \text{ is commutative with exact rows}$$

$$\text{then } \begin{array}{ccc} H_n(E) & \xrightarrow{\partial} & H_{n-1}(C) \\ \gamma_* \downarrow & & \downarrow \alpha_* \\ H_n(E') & \xrightarrow{\partial} & H_{n-1}(C') \end{array} \text{ commutes.} \quad \square$$

Useful theorems for homology with  $\mathbb{Z}$ -coefficients may now be generalized to arbitrary coefficients  $M$  in one of the following ways:

- \* Deduce from Eilenberg-Steenrod axioms
- \* Deduce from the  $\mathbb{Z}$ -version
- \* Prove in the same way as for  $\mathbb{Z}$

**Prop 9**  $H_0(X; M) \cong \bigoplus_{Z \in \pi_0(X)} \underbrace{\{[\sigma_Z \otimes m] \mid m \in M\}}_{\cong M}$ , where one chooses

$\sigma_Z: \underset{\{*\}}{\Delta^0} \rightarrow X$ ,  $\sigma(*) \in Z$  for each path-connected comp.  $Z \in \pi_0(X)$ .

**Theorem 10** (Mayer-Vietoris) If  $A, B \subseteq X$  with  $A^\circ \cup B^\circ = X$ , then there is a LES

$$\dots \rightarrow H_n(A \cap B; \mathbb{M}) \xrightarrow{\begin{pmatrix} \text{incl}_* \\ \text{incl}_* \end{pmatrix}} H_n(A; \mathbb{M}) \oplus H_n(B; \mathbb{M}) \xrightarrow{\text{incl}_* - \text{incl}_*} H_n(X; \mathbb{M}) \rightarrow H_{n-1}(A \cap B; \mathbb{M}) \rightarrow \dots$$

**Theorem 11** If  $(X, A)$  is a good pair (i.e.  $A \subseteq X$  is closed and a strong deformation retract of  $X$ ), then the projection map  $p: X \rightarrow X/A$  induces isos  $p_*: H_n(X, A; \mathbb{M}) \rightarrow H_n(X/A, A/A; \mathbb{M}) \cong \tilde{H}_n(X/A; \mathbb{M})$

**Remark 12** **Reduced homology groups**  $\tilde{H}_n(X; \mathbb{M})$  may be defined as over  $\mathbb{Z}$  coefficients for  $X \neq \emptyset$ . One has

$$\tilde{H}_n(X; \mathbb{M}) \cong H_n(X, \{x_0\}; \mathbb{M}) \underset{\text{if } n > 0}{\cong} H_n(X)$$

$$\text{and } H_0(X; \mathbb{M}) \cong \mathbb{M} \oplus \tilde{H}_0(X; \mathbb{M}).$$

**Def (AlgTop I)**  $X$  a CW-complex with cells  $e_\alpha^n$ . Let

$C_n^{\text{CW}}(X) =$  free abelian group with basis  $e_\alpha^n$  and

$$d: C_n^{\text{CW}}(X) \rightarrow C_{n-1}^{\text{CW}}(X) \text{ given by } de_\alpha^n = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where  $d_{\alpha\beta} \in \mathbb{Z}$  is the degree of

$$S^{n-1} \xrightarrow{\substack{\text{attaching} \\ \text{map of } e_\alpha^n}} X^{n-1} / (X^{n-1} \setminus e_\beta^{n-1}) \cong S^{n-1}$$

$\uparrow \qquad \qquad \uparrow$   
 $(n-1)\text{-skeleton of } X = \bigcup_{\substack{k \leq n \\ \alpha}} e_\alpha^k$

$C^{\text{CW}}(X)$  is the **cellular chain complex of  $X$**  and

$H_n^{\text{CW}}(X) := H_n(C^{\text{CW}}(X))$  the **cellular homology of  $X$** .

**Theorem 13**  $H_n^{\text{CW}}(X; \mathbb{M}) := H_n(C^{\text{CW}}(X) \otimes \mathbb{M}) \cong H_n(X; \mathbb{M})$

### ③ Calculations & the theorem of Borsuk-Ulam

**Prop 1** For all  $k \geq 0$ ,  $\tilde{H}_n(S^k; M) \cong M$  if  $n=k$ , trivial otherwise

**Three ways to prove it** (1)  $S^k$  has a CW structure with one 0-cell, one  $k$ -cell.

(2) Mayer-Vietoris with  $A = S^k \setminus e_1$ ,  $B = S^k \setminus -e_1$

(3) LES of the good pair  $(D^k, \partial D^k)$  □

**Def** Real Projective  $k$ -space  $\mathbb{RP}^k := S^k / x \sim -x$

**Prop 2**  $\ast \mathbb{RP}^k \cong (\mathbb{RP}^{k+1} \setminus \{\bar{0}\}) / x \sim \lambda x$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$

$\ast \mathbb{RP}^0 = \text{one point space}, \quad \mathbb{RP}^1 \cong S^1$

$\ast$  Alg Top I:  $H_n(\mathbb{RP}^k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & 1 \leq n \leq k-1, n \text{ odd} \\ 0 & 1 \leq n \leq k-1, n \text{ even} \\ \mathbb{Z} & n=k \text{ odd} \\ 0 & n=k \text{ even} \\ 0 & k+1 \leq n \end{cases}$

**Prop 3**  $H_n(\mathbb{RP}^k; \mathbb{Z}/2) \cong \mathbb{Z}/2$  if  $0 \leq n \leq k$  and 0 otherwise.

**Prop 4** Let  $f: Y \rightarrow X$  be a twofold covering. Then there is a LES

$$\dots \rightarrow H_n(X; \mathbb{Z}/2) \rightarrow H_n(Y; \mathbb{Z}/2) \xrightarrow{f_*} H_n(X; \mathbb{Z}/2) \rightarrow H_{n-1}(X; \mathbb{Z}/2) \rightarrow \dots$$

(a special case of the **Lyapun** LES)

**Proof** Recall that: a cont. map  $\sigma: Z \rightarrow X$  on a contractible

space  $Z$  has exactly two lifts  $\tilde{\sigma}_1, \tilde{\sigma}_2: Z \rightarrow Y$ . Here, a

**lift** is a map  $\tilde{\sigma}: Z \rightarrow Y$  so that

$$\begin{array}{ccc} & & Y \\ & \nearrow \tilde{\sigma} & \downarrow f \\ Z & \xrightarrow{\sigma} & X \end{array}$$

commutes.

Define the so-called transfer homomorphism  $T: C_n(X) \rightarrow C_n(Y)$  15

by  $T(\sigma: \Delta^n \rightarrow X) = \tilde{\sigma}_1 + \tilde{\sigma}_2$ . Check that  $T$  is a chain map.

We'll show that the short sequence of complexes

$$0 \rightarrow C(X) \otimes \mathbb{Z}/2 \xrightarrow{T} C(Y) \otimes \mathbb{Z}/2 \xrightarrow{f_c} C(X) \otimes \mathbb{Z}/2 \rightarrow 0$$

is exact. This induces the derived LES in homology (Lemma 2.9).

\*  $f_c$  surjective Lifts exist. ✓

\*  $T$  is injective. For a sing simplex  $\tau: \Delta^n \rightarrow X$ ,

let  $p_\tau: C(X) \otimes \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$  be the projection  $\sum \sigma \otimes \lambda_\sigma \mapsto \lambda_\tau$ .

$$c = \sum \sigma \otimes \lambda_\sigma \neq 0 \Rightarrow \exists \tau \text{ with } \lambda_\tau = 1 \text{ for some } \tau$$

$$\Rightarrow \lambda_{\tilde{\tau}}(T(c)) = 1 \text{ for } \tilde{\tau} \text{ a lift of } \tau \Rightarrow T(c) \neq 0. \quad \checkmark$$

\*  $\text{im}(T) = \ker f_c$ .  $f_c(c = \sum \sigma \otimes \lambda_\sigma) = 0$

$$\Leftrightarrow p_\tau(f_c(c)) = 0 \quad \forall \tau: \Delta^n \rightarrow X.$$

Since  $p_\tau(f_c(c)) = p_{\tilde{\tau}_1}(c) + p_{\tilde{\tau}_2}(c)$ , it follows that

$$f_c(c) = 0 \Leftrightarrow c = \sum_{\tau: \Delta^n \rightarrow X} \lambda_\tau (\tilde{\tau}_1 + \tilde{\tau}_2) = T\left(\sum_{\tau} \lambda_\tau \tau\right)$$

$$\Leftrightarrow c \in \text{im}(T). \quad \checkmark$$

□

**Prop 4** Let  $f: Y \rightarrow X$  be a twofold covering. Then there is a LES  
 $\dots \rightarrow H_n(X; \mathbb{Z}/2) \rightarrow H_n(Y; \mathbb{Z}/2) \xrightarrow{f_*} H_n(X; \mathbb{Z}/2) \rightarrow H_{n-1}(X; \mathbb{Z}/2) \rightarrow \dots$   
 (a special case of the Gysin LES)

**Today** For the remainder of ③:  $H_n(X, A)$  means  $H_n(X, A; \mathbb{Z}/2)$

**Prop 3**  $H_n(\mathbb{R}P^k) \cong \mathbb{Z}/2$  if  $0 \leq n \leq k$  and 0 otherwise.

**Proof** We already know this for  $n=0,1$ . So assume  $n \geq 2$ .

For the covering  $f: S^m \rightarrow \mathbb{R}P^m$ , the Gysin LES breaks into pieces:

$$0 \rightarrow H_n(\mathbb{R}P^m) \xrightarrow{\partial} H_n(\mathbb{R}P^m) \xrightarrow{T_*} H_n(S^m) \xrightarrow{f_*} H_n(\mathbb{R}P^m) \rightarrow 0$$

All homology groups are  $\mathbb{Z}/2$ -vector spaces (by Prop 2.8).

$f_*$  surjective and  $H_0(S^m) \Rightarrow H_0(\mathbb{R}P^m) \cong \mathbb{Z}/2$  or 0.

Exactness at  $H_0(S^m) \Rightarrow H_0(\mathbb{R}P^m) \cong \mathbb{Z}/2 \Rightarrow f_* = 1 \Rightarrow T_* = 0$

$\Rightarrow H_1(\mathbb{R}P^m) \cong \mathbb{Z}/2$ .

$$0 \rightarrow H_k(\mathbb{R}P^m) \xrightarrow{\partial} H_{k-1}(\mathbb{R}P^m) \rightarrow 0 \text{ if } k \notin \{0, 1, m, m+1\}$$

So,  $H_k(\mathbb{R}P^m) \cong H_{k-1}(\mathbb{R}P^m) \Rightarrow H_k(\mathbb{R}P^m) \cong \mathbb{Z}/2$  for  $k \leq m-1$

by induction.

$$0 \rightarrow H_{m+1}(\mathbb{R}P^m) \xrightarrow{\partial} H_m(\mathbb{R}P^m) \xrightarrow{T_*} H_m(S^m) \xrightarrow{f_*} H_m(\mathbb{R}P^m) \xrightarrow{\partial} H_{m-1}(\mathbb{R}P^m) \rightarrow 0$$

$\mathbb{Z}/2$

$\mathbb{Z}/2$

Since  $\mathbb{R}P^m$  has a CW-structure without  $k$ -cells for  $k \geq m+1$

$\Rightarrow H_k(\mathbb{R}P^m) = 0$  for  $k \geq m+1$ .

$\Rightarrow H_m(\mathbb{R}P^m)$  surjects onto  $\mathbb{Z}/2$ , and injects into  $\mathbb{Z}/2$

$\Rightarrow H_m(\mathbb{R}P^m) \cong \mathbb{Z}/2$ .

□

**Prop 5** The Gysin sequence from Prop 4 is natural, ie if

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \alpha \downarrow & & \downarrow \beta \\ Y' & \xrightarrow{f'} & X' \end{array}$$

commutes and  $f, f'$  are two fold coverings, then

$$\begin{array}{ccccccc} \dots & \rightarrow & H_n(X) & \xrightarrow{T_*} & H_n(Y) & \xrightarrow{f_*} & H_n(X) \xrightarrow{\partial} H_{n-1}(X) \rightarrow \dots \\ & & \downarrow \beta_* & & \downarrow \alpha_* & & \downarrow \beta_* \\ \dots & \rightarrow & H_n(X') & \xrightarrow{T'_*} & H_n(Y') & \xrightarrow{f'_*} & H_n(X') \xrightarrow{\partial} H_{n-1}(X') \rightarrow \dots \end{array}$$

commutes.

**Proof** Check that

$$\begin{array}{ccccccc} 0 \rightarrow C_n(X) \otimes \mathbb{Z}/2 & \xrightarrow{T} & C_n(Y) \otimes \mathbb{Z}/2 & \xrightarrow{f_c} & C_n(X) \otimes \mathbb{Z}/2 \rightarrow 0 \\ \beta_c \downarrow & & \alpha_c \downarrow & & \beta_c \downarrow \\ 0 \rightarrow C_n(X') \otimes \mathbb{Z}/2 & \xrightarrow{T} & C_n(Y') \otimes \mathbb{Z}/2 & \xrightarrow{f'_c} & C_n(X') \otimes \mathbb{Z}/2 \rightarrow 0 \end{array}$$

commutes, then use Lemma 2.8. □

**Borsuk-Ulam Theorem**  $f: S^m \rightarrow \mathbb{R}^m$  continuous  $\Rightarrow$

$$\exists x \in S^m: f(x) = f(-x).$$

**Proof** If no such  $x$  exists, let  $g: S^m \rightarrow S^{m-1}$ ,

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}. \text{ Then } g(-x) = -g(x).$$

This contradicts the following theorem. □

**Theorem 6** There is a cont. map  $g: S^m \rightarrow S^{m-1}$  with  
and  $g(-x) = -g(x) \iff n \leq m$ .

**Proof** If  $n \leq m$ , the embedding  $i: (x_1, \dots, x_{n+1})$

$$\mapsto (x_1, \dots, x_{n+1}, 0, \dots, 0) \text{ satisfies } i(-x) = -i(x).$$

For the other direction, assume  $n > m \geq 1$  and

let such a  $g$  be given. If  $p_m(x) = p_m(y)$ , then  $p_m \circ g(x) = p_m \circ g(y)$ . 18

Because the covering  $p_m$  is a quotient map, there is  $h: \mathbb{R}P^m \rightarrow \mathbb{R}P^m$  s.t.

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^n \\ p_m \downarrow & \searrow^{p_m \circ g} & \downarrow p_m \\ \mathbb{R}P^n & \xrightarrow{h} & \mathbb{R}P^m \end{array}$$

commutes.

Now, apply Prop 5 (naturality of the Gysin sequence) to the pieces of the Gysin LES (see proof of Prop 3):

$$\begin{array}{ccccccc} 0 \rightarrow H_k(\mathbb{R}P^n) & \xrightarrow{\text{iso}} & H_{k-1}(\mathbb{R}P^n) & \rightarrow & 0 \\ & \downarrow h_{*,k} & & \downarrow h_{*,k-1} & \\ 0 \rightarrow H_k(\mathbb{R}P^m) & \xrightarrow{\text{iso}} & H_{k-1}(\mathbb{R}P^m) & \rightarrow & 0 \end{array}$$

commutes for  $1 \leq k \leq m-1$ . Also,  $h_{*,0}$  is iso because  $\mathbb{R}P^n, \mathbb{R}P^m$  path-connected  $\Rightarrow h_{*,1}$  iso  $\Rightarrow h_{*,2}$  iso  $\Rightarrow \dots \Rightarrow h_{*,m-1}$  iso.

$$\begin{array}{ccccccc} \xrightarrow{\text{iso}} H_m(\mathbb{R}P^n) & \xrightarrow{0} & H_m(S^n) & \xrightarrow{0} & H_m(\mathbb{R}P^m) & \xrightarrow{\text{iso}} & H_{m-1}(\mathbb{R}P^m) \rightarrow 0 \\ \downarrow \text{iso} & & \downarrow & & \downarrow \text{iso} & & \downarrow \text{iso} \\ 0 \rightarrow H_m(\mathbb{R}P^m) & \xrightarrow{\text{iso}} & H_m(S^m) & \xrightarrow{0} & H_m(\mathbb{R}P^m) & \xrightarrow{\text{iso}} & H_{m-1}(\mathbb{R}P^m) \rightarrow 0 \end{array}$$

$\mathbb{Z}/2$ 
 $0$ 
 $\mathbb{Z}/2$ 
 $\mathbb{Z}/2$ 
 $\mathbb{Z}/2$

$\mathbb{Z}/2$ 
 $\mathbb{Z}/2$ 
 $\mathbb{Z}/2$ 
 $\mathbb{Z}/2$ 
 $\mathbb{Z}/2$

Contradiction!

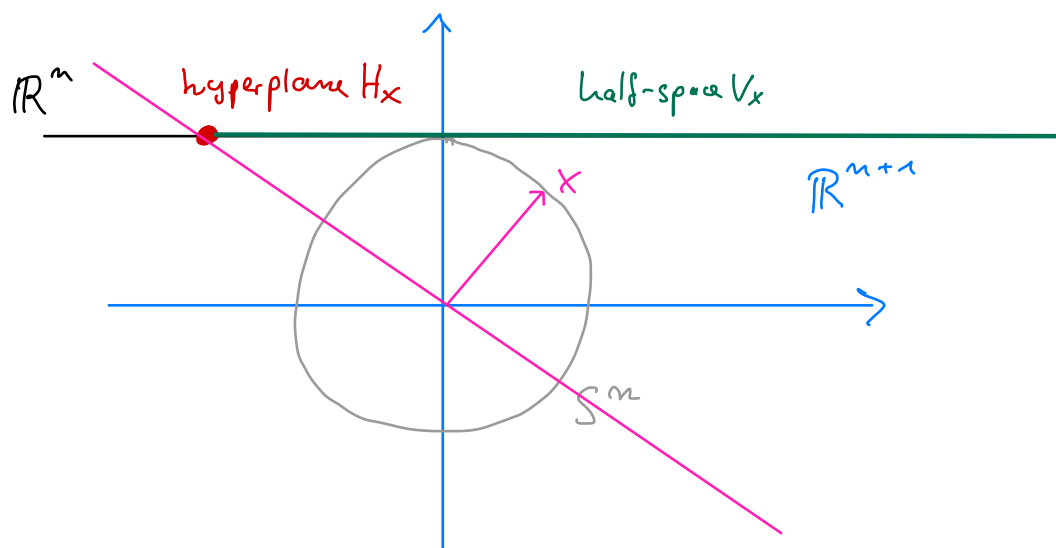
□



The Ham Sandwich Theorem  $A_1, \dots, A_m \subseteq \mathbb{R}^n$  Lebesgue-measurable & bounded

$\Rightarrow \exists$  hyperplane in  $\mathbb{R}^n$  cutting each  $A_i$  in half by volume.

Proof Identify  $\mathbb{R}^n$  with  $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$ .



For  $x \in S^n$ , let  $H_x = \mathbb{R}^n \times \{1\} \cap \{y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle = 0\}$

$V_x = \mathbb{R}^n \times \{1\} \cap \{y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle \geq 0\}$

Let  $f: S^n \rightarrow \mathbb{R}^m$ ,  $f_i(x) = \text{vol}(V_x \cap A_i)$ .

$f$  is continuous since the  $A_i$  are bounded.

Borsuk-Ulam  $\Rightarrow \exists x \in S^n: f(x) = f(-x)$

$\Rightarrow \text{vol}(V_x \cap A_i) = \text{vol}(V_{-x} \cap A_i) = \text{vol}(A_i \setminus V_x)$

$\Rightarrow H_x$  cuts all  $A_i$  in half.

□

#### ④ The Universal Coefficient Theorem for Homology

8 March

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**The Splitting Lemma** For a SES  $0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$  of abelian groups, the following are equivalent:

(1) There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & P \rightarrow 0 \\ & & \downarrow \text{id}_M & & \downarrow \text{iso} & & \downarrow \text{id}_P \\ 0 & \rightarrow & M & \xrightarrow{\text{incl}} & M \oplus P & \xrightarrow{\text{proj}} & P \rightarrow 0 \end{array}$$

(2)  $\exists i: P \rightarrow N$  with  $g \circ i = \text{id}_P$ .

(3)  $\exists r: N \rightarrow M$  with  $r \circ f = \text{id}_M$ .

SES satisfying these conditions are called **Split**.

**UCT for Homology** Let  $C$  be a chain complex of free abelian groups.

Let  $M$  be an abelian group.

(1) For all  $n$ , there is a split SES of abelian groups:

$$[x] \otimes_m \mapsto [x \otimes m]$$

$$0 \rightarrow H_n(C) \otimes M \rightarrow H_n(C; M) \rightarrow \text{Tor}(H_{n-1}(C), M) \rightarrow 0$$

(2) This SES is natural, i.e. for a chain map  $f: C \rightarrow C'$

$$0 \rightarrow H_n(C) \otimes M \rightarrow H_n(C; M) \rightarrow \text{Tor}(H_{n-1}(C), M) \rightarrow 0$$

$$\downarrow f_* \otimes \text{id}_M$$

$$\downarrow f_*$$

$$\downarrow \text{Tor}(f_*, \text{id}_M)$$

$$0 \rightarrow H_n(C') \otimes M \rightarrow H_n(C'; M) \rightarrow \text{Tor}(H_{n-1}(C'), M) \rightarrow 0$$

Commutates.

(3) There is no natural choice of splitting maps  
→ Exercise 2.4

Correction 12 March

In the lecture it was erroneously claimed that "or" suffices here

**Remark 1**  $\text{Tor}(N, M)$  will be defined for all abelian groups  $N, M$ .

We will show that for if  $M$  and  $N$  are finitely generated, then

$$\text{Tor}(N, M) \cong T(N) \otimes T(M), \text{ where}$$

$$T(N) = \{x \in N \mid \exists \lambda \in \mathbb{Z} \setminus \{0\} : \lambda x = 0\} \text{ is the torsion subgroup of } N.$$

**Remark** The UCT implies that homology with any coefficients can be read off homology with  $\mathbb{Z}$  coefficients, i.e.  $\mathbb{Z}$  coefficients are "universal". However, for a cont. map  $f$ ,  $f_*$  on  $H(-; M)$  is in general not determined by  $f_*$  on  $H(-; \mathbb{Z})$ .

→ Exercise 2.4

**Example 2** For  $\mathbb{RP}^3$ ,  $H_0 \cong \mathbb{Z}$ ,  $H_1 \cong \mathbb{Z}/2$ ,  $H_2 \cong 0$ ,  $H_3 = \mathbb{Z}$

UCT for  $M = \mathbb{Z}/2$ :

$$0 \rightarrow \underbrace{H_1(\mathbb{RP}^3) \otimes \mathbb{Z}/2}_{\mathbb{Z}/2} \rightarrow \underbrace{H_1(\mathbb{RP}^3; \mathbb{Z}/2)}_{\mathbb{Z}/2} \rightarrow \underbrace{\text{Tor}(H_0(\mathbb{RP}^3), \mathbb{Z}/2)}_{0} \rightarrow 0$$

$$0 \rightarrow \underbrace{H_2(\mathbb{RP}^3) \otimes \mathbb{Z}/2}_0 \rightarrow \underbrace{H_2(\mathbb{RP}^3; \mathbb{Z}/2)}_{\mathbb{Z}/2} \rightarrow \underbrace{\text{Tor}(H_1(\mathbb{RP}^3), \mathbb{Z}/2)}_{\mathbb{Z}/2} \rightarrow 0$$

**Reminders**  $M$  finitely generated abelian group  $\Rightarrow$

$$M = M^a \oplus \bigoplus_{\substack{p \text{ prime} \\ r \geq 1}} (\mathbb{Z}/p^r)^{b_{p,r}} \quad \text{with } a, b_{p,r} \text{ uniquely determined.}$$

$a$  is called the **rank of  $M$** , written **rk  $M$**  or **rank  $M$** .

**Prop 3** Assume  $\bigoplus_n H_n(X)$  is finitely generated. Let  $\mathbb{F}$  be a field of characteristic  $p$ .

$$\dim_{\mathbb{F}} H_n(X; \mathbb{F}) = \begin{cases} \text{rank } H_n(X) & \text{if } p=0 \\ \text{rank } H_n(X) + \# \mathbb{Z}/p^r\text{-summands of } H_n(X) + \# \mathbb{Z}/p^r\text{-summands of } H_{n-1}(X) & \text{else} \end{cases}$$

**Proof** UCT  $\Rightarrow H_n(X; \mathbb{F}) \cong H_n(X) \otimes \mathbb{F} \oplus \text{Tor}(H_{n-1}(X), \mathbb{F})$

Correction 12 March

The Proposition is true, but the proof doesn't work in general since  $\mathbb{F}$  need not be finitely generated.

We'll need to understand Tor better first to prove Prop 3

$$\cong T(H_{n-1}(X)) \otimes T(\mathbb{F})$$

by Remark 1

Now use  $T(\mathbb{F}) = \begin{cases} 0 & \text{if } p=0 \\ \mathbb{F} & \text{else} \end{cases}$

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and  $\mathbb{Z}/m \otimes \mathbb{F} \cong \mathbb{F}/m \cong \begin{cases} 0 & p|m \\ \mathbb{F} & \text{else} \end{cases} \quad \square$

**Prop 4** Let  $X$  be a space s.t.  $H_n(X) \cong 0$  for sufficiently large  $n$ , and  $H_n(X)$  finitely generated for all  $n$ . Then

$$\sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{F}}(H_n(X; \mathbb{F})) \in \mathbb{Z}$$

does not depend on the choice of a field  $\mathbb{F}$ . This integer is called the **Euler characteristic** of  $X$ , written  $\chi(X)$ .

**Proof** Note that  $(\# \mathbb{Z}/p^r\text{-summands of } H_n(X))$  appears as summand in  $\dim H_n(X; \mathbb{F})$  and in  $\dim H_{n+1}(X; \mathbb{F})$ . So, this cancels in  $\chi$  due to opposite signs.  $\square$

To prove the UCT, we need a fundamental tool of homological algebra. Let  $R$  be a commutative ring.

**Def** A **free resolution** of an  $R$ -Module  $M$  is a LES

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

where the  $F_i$  are free  $R$ -Modules.

To prove the UCT, we need a fundamental tool of homological algebra. Let  $R$  be a commutative ring.

**Def** A **free resolution**  $F$  of an  $R$ -Module  $M$  is a LES

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

where the  $F_i$  are free  $R$ -Modules.

Today

13 March

Note that  $\dots \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow 0$  is a chain complex. It is called **deleted resolution**, denoted  $F^M$ , with  $H_0(F^M) \cong M$ ,  $H_n(F^M) \cong 0$  for  $n \neq 0$ . Understanding  $H_n(F^M; N)$  is a special case of understanding  $H_n(C; N)$  for all complexes!

$$\begin{array}{ccccccc} \text{Ex} & \text{For } R = \mathbb{Z}: & \dots \rightarrow 0 & \rightarrow 0 & \rightarrow \mathbb{Z} & \xrightarrow{3} \mathbb{Z} & \rightarrow \mathbb{Z}/3 \rightarrow 0 \\ & & & & & & \\ & & & & \dots & 0 \rightarrow \mathbb{Z} & \rightarrow \mathbb{Z} \rightarrow 0 \\ & & & & & & \\ & & & & \dots & 0 \rightarrow \mathbb{Z} & \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z} \rightarrow 0 \\ & & & & & & \\ & & & & & & ? \rightarrow \mathbb{Q} \rightarrow 0 \end{array}$$

**Prop 5** Every module has a free resolution.

**Lemma 6** For every module  $M$  there exists a free module  $F$  with a surjection  $p: F \rightarrow M$ .

**Proof**

$$F := \bigoplus_{x \in M} R_x \quad \text{with } R_x \cong R. \quad F \text{ is free (with basis}$$

indexed by  $M$ ) and  $p: F \rightarrow M$ ,  $R_x \ni 1 \mapsto x$  is surjective.  $\square$

**Proof of Prop 5** Pick  $d_0: F_0 \rightarrow M$  with  $d_0$  surjective,  $F_0$  free.

Pick  $d'_1: F_1 \rightarrow \ker d_0$  with  $d'_1$  surjective,  $F_1$  free and let

$$d_1: F_1 \rightarrow F_0, \quad d_1 = (\ker d_0 \hookrightarrow F_0) \circ d'_1.$$

Pick  $d'_2: F_2 \rightarrow \ker d_1$  with  $d'_2$  surjective,  $F_2$  free ... etc.  $\square$

**Thm 7** Every subgroup of a free abelian group is free abelian.

Proof using Zorn's Lemma (see eg Lang "Algebra" Appendix 2 §2)

**Prop 8** For  $R = \mathbb{Z}$ : Every abelian group  $M$  has a free resolution of length 1, ie

$$0 \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

**Proof** Pick  $d_0: F_0 \rightarrow M$  with  $d_0$  surjective,  $F_0$  free. By Thm ,  $\ker d_0$  is free. So let  $F_1 = \ker d_0$ , and  $d_1$  the inclusion.  $\square$

**Prop 9** ("Comparison Thm", "Fundamental Thm of Homological Algebra")

(1) If  $f: M \rightarrow N$  is  $R$ -linear and  $F, G$  are free resolutions of  $M, N$ , then  $f$  may be extended to a chain map  $\hat{f}: F^M \rightarrow G^N$ , ie

$$\begin{array}{ccccccc} \cdots & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \rightarrow 0 \\ & \downarrow \hat{f}_1 & & \downarrow \hat{f}_0 & & \downarrow f & \\ \cdots & G_1 & \xrightarrow{e_1} & G_0 & \xrightarrow{e_0} & N & \rightarrow 0 \end{array}$$

(2)  $\hat{f}$  is unique up to homotopy.

(3)  $F, G$  free resolutions of  $M \Rightarrow$  The unique chain map  $F^M \rightarrow G^M$  extending  $\text{id}_M$  is a homotopy equivalence.

Proof (1)

$$\begin{array}{ccc} & F_0 & \\ \exists \hat{f}_0 \downarrow & \searrow f \circ d_0 & \\ G_0 & \xrightarrow{e_0} & N \end{array}$$

Since  $e_0$  surjective and  $F_0$  free,

there is  $\hat{f}_0 : F_0 \rightarrow G_0$  making the diagram commute (proof: for each basis element

$b$  of  $F_0$ , pick  $\hat{f}_0(b)$  such that  $e_0(\hat{f}_0(b)) = f(d_0(b))$ ).

$$\begin{array}{ccc} F_1 & & \\ \searrow \hat{f}_0 \circ d_1 & & \\ G_1 & \xrightarrow{e_1} & G_0 \end{array}$$

$$f(d_0(d_1(x))) = 0 \quad \forall x \Rightarrow e_0(\hat{f}_0(d_1(x))) = 0 \quad \forall x$$

$$\Rightarrow \text{im } \hat{f}_0 \circ d_1 \subseteq \ker e_0 = \text{im } e_1.$$

$\Rightarrow \exists \hat{f}_1 : F_1 \rightarrow G_1$  making the diagr commute etc.

(2) Let two such chain maps be given, and let  $g$  be their difference.

Then:

$$\begin{array}{ccccccc} \dots & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & \downarrow g_2 & \swarrow h_1 & \downarrow g_1 & \swarrow h_0 & \downarrow g_0 & & \downarrow 0 & & \\ \dots & G_2 & \xrightarrow{e_2} & G_1 & \xrightarrow{e_1} & G_0 & \xrightarrow{e_0} & N & \longrightarrow & 0 \end{array}$$

commutes.  $0 = 0 \circ d_0 = e_0 \circ g_0 \Rightarrow \text{im } g_0 \subseteq \ker e_0 = \text{im } e_1.$

$\Rightarrow \exists h_0$  with  $e_1 \circ h_0 = g_0$

$$e_1 \circ (g_1 - h_0 \circ d_1) = e_1 \circ g_1 - g_0 \circ d_1 = 0$$

$\Rightarrow \exists h_1$  with  $e_2 \circ h_1 = g_1 - h_0 \circ d_1$  etc.

(3)  $F, G$  free res. of  $M \Rightarrow \exists$  chain maps  $\hat{f} : F^M \rightarrow G^M$  and

$\hat{g} : G^M \rightarrow F^M$  that extend  $\text{id}_M : M \rightarrow M \Rightarrow \hat{g} \circ \hat{f} : F^M \rightarrow F^M$

and  $\hat{f} \circ \hat{g} : G^M \rightarrow G^M$  extend  $\text{id}_M$ , but so do  $\text{id}_{F^M}, \text{id}_{G^M}$

$\Rightarrow$  By uniqueness,  $\hat{g} \circ \hat{f} \simeq \text{id}_{F^M}, \hat{f} \circ \hat{g} \simeq \text{id}_{G^M}.$

□

**Def** Let  $M, N$  be  $R$ -Modules, and  $F$  a free resolution of  $M$ , then  $\text{Tor}_n(M, N) := H_n(F^\bullet; N)$  for  $n \geq 0$ .

**Proof** that  $\text{Tor}$  does not depend on choice of  $F$ :  $F, G$  free res. of  $M$   
 $\Rightarrow F^\bullet \simeq G^\bullet \Rightarrow F^\bullet \otimes N \simeq G^\bullet \otimes N$  (Cor (2.7 (3)))  $\Rightarrow$   
 $H_n(F^\bullet; N) \cong H_n(G^\bullet; N)$ .  $\square$

**Remark 10** Over  $R = \mathbb{Z}$ ,  $\text{Tor}_n(M, N) = 0 \ \forall n \geq 2$  since  $M$  has a free res. of length 1 (Prop 8). So we write  $\text{Tor}(M, N) := \text{Tor}_1(M, N)$ .

**Lemma 11**  $f: M \rightarrow N$   $R$ -linear,  $P$   $R$ -module  $\Rightarrow$   
 $(\text{Coker } f) \otimes P \cong \text{Coker}(f \otimes \text{id}_P)$ . **Proof** Exercise.

**Proof of the UCT** (1) Constructing the SES

$$\underbrace{B_n = \text{im } d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \text{ker } d_n}_{n\text{-cycles}}$$

Make  $B_n, Z_n$  into chain complexes, taking  $d$  as differential.

There is a SES of chain complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_{n+1} & \xrightarrow{\text{incl}} & C_{n+1} & \xrightarrow{d} & B_n \longrightarrow 0 \\ & & \downarrow \scriptstyle 0 & & \downarrow d & & \downarrow \scriptstyle 0 \\ 0 & \longrightarrow & Z_n & \xrightarrow{\text{incl}} & C_n & \xrightarrow{d} & B_{n-1} \longrightarrow 0 \\ & & & & \vdots & & \end{array}$$



$$\underbrace{B_n = \operatorname{im} d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \ker d_n}_{n\text{-cycles}}$$

Make  $B_n, Z_n$  into chain complexes, taking 0 as differential.

There is a SES of chain complexes:

$$\begin{array}{ccccccc} & & & \vdots & & & \\ 0 & \longrightarrow & Z_{n+1} & \xrightarrow{\text{incl}} & C_{n+1} & \xrightarrow{d} & B_n \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow d & & \downarrow 0 \\ 0 & \longrightarrow & Z_n & \xrightarrow{\text{incl}} & C_n & \xrightarrow{d} & B_{n-1} \longrightarrow 0 \\ & & & \vdots & & & \end{array}$$

$B_n$  free by Thm 7  $\Rightarrow$  each row splits  $\Rightarrow$  tensoring with  $M$  preserves exactness (Exercise). The  $\text{SES} \otimes M$  induces a LES:

$$\cdots \rightarrow B_n \otimes M \xrightarrow{\tau} Z_n \otimes M \rightarrow \frac{\ker d_n \otimes \text{id}_M}{\operatorname{im} d_{n+1} \otimes \text{id}_M} \xrightarrow{\text{incl} \otimes \text{id}_M} B_{n-1} \otimes M \rightarrow Z_{n-1} \otimes M \rightarrow \cdots$$

$$\Rightarrow \text{SES} \quad 0 \rightarrow \underbrace{H_n(C) \otimes M}_{\cong \text{Coker } \tau \text{ by Lemma 11}} \rightarrow H_n(C; M) \rightarrow \ker \text{incl} \otimes \text{id}_M \rightarrow 0$$

There is a SES

$$0 \rightarrow B_{n-1} \xrightarrow{\text{incl}} Z_{n-1} \rightarrow H_{n-1}(C) \rightarrow 0$$

which is a free resolution of  $H_{n-1}(C)$ . So

$$\ker \text{incl} \otimes \text{id}_M \cong \operatorname{Tor}(H_{n-1}(C), M).$$

(1) The SES splits  $C_n$  free  $\Rightarrow \exists p_n: C_n \rightarrow Z_n$  st.

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$$\text{incl} \circ p_n = \text{id}_{Z_n}.$$

Correction 5 April

$p: C \rightarrow Z$  is in general not

a chain map! (Indeed,  $p$  chain map  $\Rightarrow$  differential of  $C$  is zero). Proceed instead as follows: Let  $\pi_n: Z_n \rightarrow H_n(C) = Z_n/B_n$  be the projection. Then  $\pi_n \circ p_n$  is a map  $C_n \rightarrow H_n(C)$ , and this is a chain map when one considers  $H_n(C)$  as complex with zero differential (since for  $x \in C_n: d_n(x) \in B_{n-1} \subseteq Z_{n-1}$ , so  $p_{n-1}(d_n(x)) = d_n(x)$  and  $\pi_{n-1}(p_{n-1}(d_n(x))) = [d_n(x)] = 0$ ). Thus  $(\pi_n \circ p_n) \otimes \text{id}_M: C_n \otimes M \rightarrow H_n(C) \otimes M$  is also a chain map, inducing a map  $H_n(C; M) \xrightarrow{q} H_n(C) \otimes M$  on homology. To see that  $q$  is a splitting map, check that  $q([x \otimes m]) = [x] \otimes m$  for all  $x \in Z_n$  and  $m \in M$ .

(2) Naturality (Sketch)

$$f: C \rightarrow C' \text{ chain map} \Rightarrow f(Z) \subseteq Z', f(B) \subseteq B'.$$

So  $f$  induces a map between the SES of chain complexes  $0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0$  and  $0 \rightarrow Z'_n \rightarrow C'_n \rightarrow B'_{n-1} \rightarrow 0$ , also after  $\otimes M$ , and so also between the associated LES, and so also between the SES in the UCT.

(3) Unnaturalness of splitting: Exercise 2.4

Prop 12  $\text{Tor}_0(M, N) \cong M \otimes N$

Proof  $\dots \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow 0$  deleted free res of  $M$ .

$$\begin{aligned} \Rightarrow \text{Tor}_0(M, N) &= \text{Coker}(d_1 \otimes \text{id}_N) \cong \text{Coker}(d_1) \otimes N \\ &= H_0(F^\bullet) \otimes N = M \otimes N \end{aligned}$$

□

Remark 13 For  $f: M \rightarrow M'$ ,  $g: N \rightarrow N'$ , one may set

$$\text{Tor}_n(f, g): \text{Tor}_n(M, N) \rightarrow \text{Tor}_n(M', N')$$

to be given by  $(\hat{f} \otimes g)_*$ . Fixing one argument then makes  $\text{Tor}_n$  into an additive functor  $R\text{-Mod} \rightarrow R\text{-Mod}$ .

**Prop 14** Let  $A, B, C$  be abelian groups.

$$(1) B \text{ free} \Rightarrow \text{Tor}(A, B) \cong 0$$

(2) If  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is exact, then

$$0 \rightarrow \text{Tor}(D, A) \rightarrow \text{Tor}(D, B) \rightarrow \text{Tor}(D, C) \rightarrow 0$$

$$\hookrightarrow D \otimes A \rightarrow D \otimes B \rightarrow D \otimes C \rightarrow 0$$

is exact.

$$(3) \text{Tor}(A, B) \cong \text{Tor}(B, A).$$

$$(4) B \text{ torsion-free} \Rightarrow \text{Tor}(A, B) \cong 0$$

$$(5) \text{Tor}(A, B) \cong \text{Tor}(\text{T}(A), \text{T}(B)).$$

$$(6) \text{Tor}(\mathbb{Z}/m, A) \cong \{x \in A \mid mx = 0\}$$

$$(7) \text{Tor}(A \oplus B, C) \cong \text{Tor}(A, C) \oplus \text{Tor}(B, C)$$

$$(8) \text{Tor}(A, B) \cong \text{T}(A) \otimes \text{T}(B) \text{ if } A \text{ and } B \text{ are f.g.}$$

**Proof** (1)  $0 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$  free res of  $A \Rightarrow$

$$0 \rightarrow F_1 \otimes B \rightarrow F_0 \otimes B \rightarrow A \otimes B \rightarrow 0 \text{ is exact} \Rightarrow \text{Tor}(A, B) \cong 0.$$

(2) Pick free res  $0 \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow D \rightarrow 0$

$$\Rightarrow \begin{array}{ccccccc} 0 & \rightarrow & F_1 & \xrightarrow{\text{id}_F \otimes f} & F_1 & \xrightarrow{\text{id}_F \otimes g} & F_1 \otimes C \rightarrow 0 \\ & & \downarrow d_1 \otimes \text{id}_A & & \downarrow d_1 \otimes \text{id}_B & & \downarrow d_1 \otimes \text{id}_C \\ 0 & \rightarrow & F_0 & \xrightarrow{\text{id}_F \otimes f} & F_0 & \xrightarrow{\text{id}_F \otimes g} & F_0 \otimes C \rightarrow 0 \end{array}$$

commutes and has exact rows. It is a SES of chain complexes!

(Each complex made of two groups). The associated LES in homology is the desired sequence. ✓

(3) Apply (1) to a free res  $0 \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow B \rightarrow 0$

$\leadsto$  LES

$$0 \rightarrow \underbrace{\text{Tor}(A, F_1)}_{=0 \text{ because } F_1 \text{ free}} \rightarrow \underbrace{\text{Tor}(A, F_0)}_{=0 \text{ because } F_0 \text{ free}} \rightarrow \text{Tor}(A, B) \rightarrow 0$$

$$\hookrightarrow A \otimes F_1 \xrightarrow{\text{id}_A \otimes d_1} A \otimes F_0 \rightarrow A \otimes B \rightarrow 0$$

$\Rightarrow \text{Tor}(A, B) \cong \ker(\text{id}_A \otimes d_1) = \text{Tor}(B, A)$  by def of  $\text{Tor}$ ,  
using  $A \otimes B \cong B \otimes A$ . ✓

(4) Pick free res  $0 \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \rightarrow 0$ .

It's enough to show that  $F_1 \otimes B \rightarrow F_0 \otimes B$  is injective.

So let  $\alpha \in F_1 \otimes B$  with  $d_1 \otimes \text{id}_B(\alpha) = 0$  be given. To show:  $\alpha = 0$ .

Claim There is a f.g. subgroup  $B' \subseteq B$  with  $\alpha \in B'$  and  $d_1 \otimes \text{id}_{B'}(\alpha) = 0$ .

Pf that Claim  $\Rightarrow \alpha = 0$   $B$  torsion-free  $\Rightarrow B'$  torsion-free.  $B'$  torsion-free and f.g.

$\Rightarrow B'$  free by classification of f.g. ab. groups. We already know that tensoring with a free module is exact  $\Rightarrow d_1 \otimes \text{id}_{B'}$  injective  $\Rightarrow \alpha = 0$ . ✓

Pf of Claim Use construction of  $\otimes$ :  $F_0 \otimes B \cong$  free module  $U_{F_0, B}$  with basis

$F_0 \times B$  modulo submodule  $I_{F_0, B} \subseteq U$  generated by

$$\begin{aligned} &(\lambda x + x', y) - \lambda(x, y) - (x', y) \\ &(x, \lambda y + y') - \lambda(x, y) - (x, y') \end{aligned} \quad (*)$$

Write  $\alpha = \sum_{i=1}^m f_i \otimes b_i$ . Then  $d_1 \otimes \text{id}_B(\alpha) = 0 \Leftrightarrow \sum d_1(f_i) \otimes b_i = 0$

$$\Leftrightarrow \sum_{i=1}^m (d_1(f_i), b_i) = \sum_{j=1}^k \text{elements of the form } (*) \in I_{F_0, B}$$

Let  $B' \subseteq B$  be generated by  $b_1, \dots, b_m$  and all elements of  $B$  appearing in the sum on the RHS. Then  $\alpha \in F_1 \otimes B'$ , and

$$d_1 \otimes \text{id}_{B'}(\alpha) = 0$$

✓

the following proofs were skipped in the lecture

(5) Apply (2) to the SES  $0 \rightarrow T(B) \rightarrow B \rightarrow B/T(B) \rightarrow 0$ :

$$0 \rightarrow \text{Tor}(A, T(B)) \rightarrow \text{Tor}(A, B) \rightarrow \text{Tor}(A, B/T(B)) \rightarrow \dots$$

0 by (4) since

$B/T(B)$  torsion-free

$\Rightarrow \text{Tor}(A, T(B)) \cong \text{Tor}(A, B)$ . Now use (3) and repeat the argument. ✓

(6)  $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n \rightarrow 0$  is a free res of  $\mathbb{Z}/n$ .

$$\Rightarrow \text{Tor}(\mathbb{Z}/n, A) \cong \ker(A \xrightarrow{n} A) = \{x \in A \mid nx = 0\} \quad \checkmark$$

(7) 
$$\left. \begin{array}{l} 0 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \\ 0 \rightarrow G_1 \rightarrow G_0 \rightarrow B \rightarrow 0 \end{array} \right\} \text{ free res.}$$

$$\Rightarrow 0 \rightarrow F_1 \oplus G_1 \rightarrow F_0 \oplus G_0 \rightarrow A \oplus B \rightarrow 0 \text{ free res}$$

$$\text{Now } \text{Tor}(A \oplus B, C) \cong \ker((F_1 \oplus G_1) \otimes C \rightarrow (F_0 \oplus G_0) \otimes C)$$

$$\cong \ker(F_1 \otimes C \rightarrow F_0 \otimes C)$$

$$\oplus \ker(G_1 \otimes C \rightarrow G_0 \otimes C)$$

$$\cong \text{Tor}(A, C) \oplus \text{Tor}(B, C) \quad \checkmark$$

(8) Using (7), (3), (1) and the classification of f.g. ab groups, it is enough to check this for  $A \cong \mathbb{Z}/a$ ,  $B \cong \mathbb{Z}/b$ .

This will be an Exercise on Sheet 3. ✓

□

## ⑤ Cohomology

**Goal** Dualize the singular chain complex, i.e. apply  $\text{Hom}(-, \mathbb{Z})$  (or  $\text{Hom}(-, M)$  for any abelian group  $M$ )  $\rightarrow$  cochain complex with cohomology. Why? Cohomology ...

\* ... has more structure than a homology (it is a ring!)

\* ... may arise in a natural way from geometric applications

**Def** A **cochain complex**  $C$  over a commutative ring  $R$  is a collection  $C^n$  of  $R$ -modules for  $n \in \mathbb{Z}$  called **cochain modules**,  $R$ -linear maps  $d^n: C^n \rightarrow C^{n+1}$  with  $d^{n+1} \circ d^n = 0$  called **differentials**.

The  **$n$ -th cohomology module** of  $C$  is

$$H^n(C) = \frac{\ker d^n}{\text{im } d^{n-1}}$$

$n$ -cocycles  
 $n$ -coboundaries

A **cochain map**  $f: C \rightarrow D$  is a collection of  $R$ -linear  $f^n: C^n \rightarrow D^n$  s.t.  $f^{n+1} \circ d_C^n = d_D^n \circ f^n \forall n$ .

$f, g: C \rightarrow D$  are **homotopic**, written  $f \simeq g$ , if  $\exists$  a **homotopy**  $h: C \rightarrow D$ , i.e. a collection of  $R$ -linear  $h^n: C^n \rightarrow D^{n-1}$ ,

$$\text{s.t. } f_n - g_n = d_D^{n-1} \circ h_n + h_{n+1} \circ d_C^n$$

**Remark 1**  $C$  cochain complex

$\Leftrightarrow D$  with  $D_n = C^{-n}$ ,  $d_n^D = d_C^{-n}$  is a chain complex

Under this 1:1-correspondence, cohomology  $\Leftrightarrow$  homology,

cochain maps  $\Leftrightarrow$  chain maps, homotopies  $\Leftrightarrow$  homotopies etc.

So everything that is true for chain complexes also holds

true mutatis mutandis for cochain complexes, e.g. Prop 2.

Prop 2 (1)  $f: C \rightarrow D$  a cochain map  $\Rightarrow$

$f^*: H^n(C) \rightarrow H^n(D)$ ,  $f^*([x]) = [f(x)]$  is a well-def.  $R$ -homom.

(2)  $H^n(-)$  is an additive functor

$$\underline{\text{CoCh}}(R) \longrightarrow R\text{-Mod}$$

category of cochain complexes over  $R$ , cochain maps

(3)  $f \simeq g \Rightarrow f^* \simeq g^*$ .

No proof

Prop 3 If  $F: R\text{-Mod} \rightarrow R\text{-Mod}$  is a contravariant additive functor, then  $F: \text{CoCh}(R) \rightarrow \text{CoCh}(R)$  is also contravariant additive:

$$\dots C_n \xrightarrow{d_n} C_{n-1} \dots \longmapsto \dots F(C_n) \xleftarrow{F(d_n)} F(C_{n-1}) \dots$$

cochain complex  $F(C)$   
with  $F(C)^n = F(C_n)$ ,  
 $d_{F(C)}^n = F(d_C^{n-1})$

No proof

Def  $X$  top. space,  $A \subseteq X$ ,  $M$  an abelian group.

Then the cochain complex obtained from  $C_n(X, A)$  by

applying  $\text{Hom}(-, M)$  is called the singular cochain

complex of  $(X, A)$  with coefficients in  $M$ , denoted  $C^n(X, A; M)$

and its cohomology the singular cohomology of  $(X, A)$  with

coefficients in  $M$ , denoted  $H^n(X, A; M)$ . We may drop " $M$ " for  $M = \mathbb{Z}$ .

For  $f: (X, A) \rightarrow (Y, B)$  continuous, write  $f^c$  for the cochain map  $C^n(Y, B; M) \rightarrow C^n(X, A; M)$ ,

$f^c = \text{Hom}(f_c, M)$ , and  $f^*$  for the induced

homom.  $H^n(Y, B; M) \rightarrow H^n(X, A; M)$ .

**Ex 4**  $C^0(X; M) = \text{Hom}(C_0(X), M)$ . Corresponds to functions  $X \rightarrow M$ . Let  $\varphi \in C^0(X; M)$ . Then  $d^0(\varphi)$  sends  $\sigma: \Delta^1 = [0, 1] \rightarrow X$  to  $\varphi(d_1(\sigma)) = \varphi(\sigma(1)) - \varphi(\sigma(0))$ . So  $d^0(\varphi) = 0 \Leftrightarrow \varphi(\sigma(0)) = \varphi(\sigma(1)) \forall \sigma \Leftrightarrow \varphi$  constant on path-connected components. Hence

$$H^0(X; M) = \ker d^0 \cong \prod_{\pi_0(X)} M$$

note: for  $\pi_0(X)$  infinite  
 $H^0(X; \mathbb{Z}) \not\cong H_0(X; \mathbb{Z})$   
 $\prod_{\pi_0(X)} \mathbb{Z} \quad \quad \quad \bigoplus_{\pi_0(X)} \mathbb{Z}$

**Remark 5** A hands-on approach to cochains:

An  $n$ -cochain  $\varphi \in C^n(X; M)$  is a homom.  $C_n(X) \rightarrow M$ .

So  $n$ -chains correspond to functions

$$\{ \text{singular } n\text{-simplices } \sigma: \Delta^n \rightarrow X \} \longrightarrow \mathbb{Z}$$

The  $(n+1)$ -cochain  $d^n(\varphi)$  sends  $\tau: \Delta^{n+1} \rightarrow X$  to  $\varphi(d_{n+1}(\tau))$ .

So  $\varphi$  is an  $n$ -cocycle  $\Leftrightarrow \varphi$  is zero on  $n$ -boundaries  $\in B_n$ .

$\varphi$  is an  $n$ -coboundary  $\Rightarrow \varphi(\sigma)$  is determined by  $d_n(\sigma)$ .  
 $\Rightarrow \varphi$  is zero on  $n$ -cycles  $\in Z_n$

**Correction 22 April** The implication " $\Leftarrow$ " does not generally hold: there may be cochains  $\varphi$  that are zero on  $n$ -cycles, but that are not coboundaries. Indeed, this happens if  $\varphi$  is a cocycle,  $[\varphi] \neq 0 \in H^n(X; M)$ , and  $\text{ev}([\varphi]) = 0$ .

Thus: An  $n$ -cocycle  $\varphi$  induces a homom.  $C_n(X)/B_n \rightarrow M$ , by restriction it also induces a homom.

$$Z_n/B_n = H_n(X) \rightarrow M.$$

For  $n$ -coboundaries  $\varphi$ , this homom. is zero. Thus we have a homom. called the **evaluation homomorphism**

$$\text{ev}: H^n(X; M) \longrightarrow \text{Hom}(H_n(X), M)$$

which may be seen to be natural in both  $X$  and  $M$ .



## Universal Coefficient Theorem for Cohomology

Let  $C$  be a chain complex of free abelian groups and  $A$  an abelian group

(1) There is a split SES

$$0 \rightarrow \underset{\substack{\uparrow \\ \text{to be defined!}}}{\text{Ext}}(H_{n-1}(C), A) \rightarrow H^n(C; M) \xrightarrow{\text{ev}} \text{Hom}(H_n(C), A) \rightarrow 0$$

(2) These SES are natural in  $C$  and  $A$ .

(3) The splittings cannot be chosen naturally

**Def** Let  $M, N$  be  $R$ -modules, and  $F$  a free res. of  $M$ . Then let

$$\text{Ext}_R^n(M, N) := H^n(\text{Hom}(F^M, N))$$

$F^M$  unique up to hom. equiv.  $\Rightarrow$  Def of Ext independent of choice of  $F$ .

As with Tor, we have:

$$* \text{Ext}_R^0(M, N) \cong \text{Hom}(M, N).$$

\*  $\text{Ext}_R^n(A, B) = 0$  for all  $n \geq 2$ , so we write  $\text{Ext}(A, B)$  for  $\text{Ext}_R^1(A, B)$ .

For the proof of the first point, one needs:

**Lemma 6**  $M, N, P$   $R$ -modules,  $f: M \rightarrow N$   $R$ -linear

$$\Rightarrow \text{Hom}(\text{coker } f, P) \cong \ker(\text{Hom}(f, P))$$

**Proof**  $M \rightarrow N \rightarrow \text{coker } f \rightarrow 0$  exact

$$\Rightarrow 0 \rightarrow \text{Hom}(\text{coker } f, P) \rightarrow \text{Hom}(N, P) \rightarrow \text{Hom}(M, P) \text{ is exact}$$

(same argument as in Ex Sheet 1, 2b)

□

**Remark 7** \* Ext is not symmetric:  $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$   
 $\text{Ext}(\mathbb{Z}, \mathbb{Z}/m) \cong 0$

(as we shall see from Prop 8)

\* Ext can behave unexpectedly:

$$\text{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \text{uncountably-dimensional } \mathbb{Q}\text{-vector space}$$

**Prop 8** For all ab groups  $A, B, C$ , the following hold:

$$(1) \operatorname{Ext}(A \oplus B, C) \cong \operatorname{Ext}(A, C) \oplus \operatorname{Ext}(B, C)$$

$$(2) \operatorname{Ext}(A, B \oplus C) \cong \operatorname{Ext}(A, B) \oplus \operatorname{Ext}(A, C)$$

$$(3) A \text{ free} \Rightarrow \operatorname{Ext}(A, B) \cong 0.$$

$$(4) \operatorname{Ext}(\mathbb{Z}/n, A) \cong A/nA$$

Note this suffices to compute  $\operatorname{Ext}$  (f.g. group,  $A$ ).

$$(5) \operatorname{Ext}(A, B) \cong T(A) \otimes B \text{ if } A, B \text{ f.g.}$$

Compare (4), (5) to  $\operatorname{Tor}$ :  $\operatorname{Tor}(\mathbb{Z}/n, A) \cong \{x \in A \mid nx = 0\}$

$$\operatorname{Tor}(A, B) \cong T(A) \otimes T(B) \text{ for } A, B \text{ f.g.}$$

**Proof of (4)**  $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n \rightarrow 0$  free res.  $\mathbb{Z}$

$$\operatorname{Hom}(\mathbb{Z}/n, A) = 0 \leftarrow \operatorname{Hom}(\mathbb{Z}, A) \xleftarrow{n} \operatorname{Hom}(\mathbb{Z}, A) \leftarrow 0$$

$$\cong A \qquad \qquad \qquad \cong A$$

$$\Rightarrow \operatorname{Ext} = H^1 \text{ of this cochain complex} \cong A/nA \quad \square$$

**Prop 9** Let  $R$ -modules  $M, N$  be given. An extension of  $N$  by  $M$

is a SES  $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ . It is equivalent

to another extension  $0 \rightarrow N \rightarrow P' \rightarrow M \rightarrow 0$  if  $\exists f: P \rightarrow P'$  st

$$\begin{array}{ccccccc} 0 & \rightarrow & N & \rightarrow & P & \rightarrow & M \rightarrow 0 \\ & & \downarrow \text{id}_N & & \downarrow f & & \downarrow \text{id}_M \\ 0 & \rightarrow & N & \rightarrow & P' & \rightarrow & M \rightarrow 0 \end{array}$$

commutes. Five-lemma  $\Rightarrow f$  is iso. So equivalence is an equiv. rel.

One finds  $\{\text{Extensions of } N \text{ by } M\} / \text{equiv} \xleftrightarrow{1:1} \operatorname{Ext}_R^1(M, N).$

**Prop 10** Assume  $H_n(X, A)$  is f.g. for all  $n$ . Then

$$H^n(X, A; \mathbb{Z}) \cong \underbrace{F(H_n(X, A))}_{\text{free part}} \oplus T(H_{n-1}(X, A))$$

$F(B) := B/T(B)$

**Proof**  $UCT \Rightarrow H^n(X, A; \mathbb{Z}) \cong \text{Hom}(H_n(X, A), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(X, A), \mathbb{Z})$

$$\cong \text{Hom}(F(H_n(X, A)), \mathbb{Z}) \cong F(H_n(X, A))$$

$$\oplus \text{Hom}(T(H_n(X, A)), \mathbb{Z}) \cong 0$$

$$\oplus \text{Ext}(F(H_{n-1}(X, A)), \mathbb{Z}) \cong 0$$

$$\oplus \text{Ext}(T(H_{n-1}(X, A)), \mathbb{Z}) \cong T(H_{n-1}(X, A)) \quad \square$$

**Def** The **cellular cochain complex**  $C_{CW}^\bullet(X)$  of a CW-complex  $X$  is  $\text{Hom}(C_{CW}^\bullet(X), M)$ . Its cohomology  $H_{CW}^n(X; M)$  is the  **$n$ -th cellular cohomology group**.

**Thm 11**  $H_{CW}^n(X; M) \cong H^n(X; M)$ .

**Example 12**  $C_{CW}^\bullet(\mathbb{R}P^2) = 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$   
 $H_0^{CW} \cong \mathbb{Z}, H_1^{CW} \cong \mathbb{Z}/2, H_2^{CW} = 0$

Hands-on Trick:  $C$  a chain complex of f.g. free ab. groups with a chosen basis, then

$$\underbrace{(\text{Matrix of } d_n)^T}_{\text{wrt to the basis}} = \underbrace{\text{Matrix of } \text{Hom}(d_n, \mathbb{Z})}_{\text{wrt the dual basis}}$$

$$\Rightarrow C_{CW}^\bullet(\mathbb{R}P^2; \mathbb{Z}) = 0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$$\text{and } H_{CW}^0 \cong \mathbb{Z}, H_{CW}^1 \cong 0, H_{CW}^2 \cong \mathbb{Z}/2$$

# Proof of UCT (1)

There is a SES of chain complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_{n+1} & \xrightarrow{\text{incl}} & C_{n+1} & \xrightarrow{d_{n+1}} & B_n \longrightarrow 0 \\ & & \downarrow \circ & & \downarrow d_{n+1} & & \downarrow \circ \\ 0 & \longrightarrow & Z_n & \xrightarrow{\text{incl}} & C_n & \xrightarrow{d_n} & B_{n-1} \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & \vdots & & \end{array}$$

$B_n$  free by Thm 4.7  $\Rightarrow$  each row splits  $\Rightarrow$  SES of cochain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(B_{n-1}, M) & \xrightarrow{d^{n-1}} & \text{Hom}(C_n, M) & \xrightarrow{\text{incl}^*} & \text{Hom}(Z_n, M) \longrightarrow 0 \\ & & \downarrow \circ & & \downarrow d^n & & \downarrow \circ \\ 0 & \longrightarrow & \text{Hom}(B_n, M) & \xrightarrow{d^n} & \text{Hom}(C_{n+1}, M) & \xrightarrow{\text{incl}^*} & \text{Hom}(Z_{n+1}, M) \longrightarrow 0 \end{array}$$

This induces a LES

$$\begin{array}{ccccccc} & & & & & & \dots \longrightarrow \text{Hom}(Z_{n-1}, M) \\ \partial^{n-1} & \longrightarrow & \text{Hom}(B_{n-1}, M) & \longrightarrow & H^n(C; M) & \longrightarrow & \text{Hom}(Z_n, M) \\ \partial^n & \longrightarrow & \text{Hom}(B_n, M) & \longrightarrow & \dots & & \end{array}$$

Check that  $\partial^i = \text{Hom}(B_n \hookrightarrow Z_n, M)$

$\Rightarrow$  SES

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underbrace{\text{coker } \partial^{n-1}} & \longrightarrow & H^n(C; M) & \longrightarrow & \underbrace{\text{ker } \partial^n} \longrightarrow 0 \\ & & \cong \text{Ext}(H_{n-1}(C), M) & & & & \cong \text{Hom}(\text{coker } B_n \rightarrow Z_n, M) \\ & & & & & & \text{(by Lemma 6)} \\ & & & & & & \cong \text{Hom}(H_n(C), M) \end{array}$$

$\hookrightarrow$  because: free res  $0 \rightarrow B_{n-1} \rightarrow Z_{n-1} \rightarrow H_{n-1}(C) \rightarrow 0$

$\leadsto$  cochain complex  $0 \leftarrow \text{Hom}(B_{n-1}, M) \xleftarrow{\partial^{n-1}} \text{Hom}(Z_{n-1}, M)$   
with  $H^1 \cong \text{coker } \partial^{n-1}$ , and  $H^1 \cong \text{Ext}$  by def of Ext.  $\square$

**Prop 11** Singular cohomology satisfies axioms that are analogue to the Eilenberg-Steenrod axioms for homology (see ②):

Data

$H^n(-; M)$  are contravariant functors  $\{\text{Pairs of spaces}\} \rightarrow \mathbb{Z}\text{-Mod}$

There are natural connecting homom.  $\partial: H^n(A; M) \rightarrow H^{n+1}(X, A; M)$

Axioms

**Homotopy** (1)  $f \simeq g \Rightarrow f^* = g^*$

**Excision** (2)  $\bar{U} \subseteq A^0 \Rightarrow \text{incl}^*: H^n(X, A; M) \rightarrow H^n(X \setminus U, A \setminus U; M)$  is 0

**Dimension** (3)  $H^n(\{*\}; M) \cong M$  for  $n=0$ , trivial for  $n \neq 0$ .

**Additivity** (4)  $H^n(\coprod_{\alpha} X_{\alpha}; M) \xrightarrow{i} \prod_{\alpha} H^n(X_{\alpha}; M)$  is an iso,  
with  $i$  given by  $i_{\alpha} = (\text{inclusion } X_{\alpha} \rightarrow \coprod_{\alpha} X_{\alpha})^*$ .

**Exactness** (5) There are LESs

$$\dots \rightarrow H^n(X, A; M) \xrightarrow{\text{incl}^*} H^n(X; M) \xrightarrow{\text{incl}^*} H^n(A; M) \xrightarrow{\partial} H^{n+1}(X, A; M) \rightarrow \dots$$

Similarly as for Homology with coefficients, all axioms follow more or less directly from homotopy equivalences of singular chain complexes being sent to hom. equiv. of singular cochain complexes by the additive  $\text{Hom}(-, M)$  functor.

**Proof of (4)** Alg Top I:  $\sum_{\alpha} (incl_{\alpha})_c: \bigoplus_{\alpha} C_{\bullet}(X_{\alpha}) \longrightarrow C_{\bullet}(\bigsqcup_{\alpha} X_{\alpha})$

is a homotopy equivalence  $\Rightarrow$  so is

$$\begin{array}{ccc}
 \text{Hom} \left( C_{\bullet} \left( \bigsqcup_{\alpha} X_{\alpha} \right), M \right) & \xrightarrow{\text{Hom} \left( \sum (incl_{\alpha})_c, M \right)} & \text{Hom} \left( \bigoplus_{\alpha} C_{\bullet}(X_{\alpha}), M \right) \\
 \downarrow \text{iso (by def)} & & \downarrow \text{iso} \\
 & & \prod_{\alpha} \text{Hom} (C_{\bullet}(X_{\alpha}), M) \\
 & & \downarrow \text{iso (by def)} \\
 C^{\bullet} \left( \bigsqcup_{\alpha} X_{\alpha}; M \right) & \xrightarrow{\alpha\text{-component is } (incl_{\alpha})_c} & \prod_{\alpha} C^{\bullet}(X_{\alpha}; M)
 \end{array}$$

□

Further good properties of cohomology:

**Thm 12 (Mayer-Vietoris)**  $A, B \subseteq X, A^{\circ} \cup B^{\circ} = X \Rightarrow LES$

$$\dots \rightarrow H^n(X; M) \rightarrow H^n(A; M) \oplus H^n(B; M) \rightarrow H^n(A \cap B; M) \rightarrow H^{n+1}(X) \rightarrow \dots$$

**Remark 13** Understanding the connection homomorphisms in the Mayer-Vietoris - sequence:

Homology  $H_n(X) \longrightarrow H_{n-1}(A \cap B):$

Represent a homology class  $[x] \in H_n(X)$  as  $[y + z]$ , where  $y \in C_n(A)$  and  $z \in C_n(B)$ . (Here, we abuse notation and write  $y$  also for the image of  $y$  under  $C_n(A) \hookrightarrow C_n(X)$ , similarly for  $z$ .) Now send  $[x] \mapsto [dy]$ .

(since  $0 = dx = d(y+z) \Rightarrow dy = -dz$ , so  $dy \in C_{n-1}(A \cap B)$ , again abusing notation).

see Hatcher p. 150

A similar understanding for cohomology is more complicated. The following wasn't discussed in the lecture.

Cohomology  $H^n(A \cap B) \longrightarrow H^{n+1}(X)$ :

Extend a cohomology class  $[\varphi] \in H^n(A \cap B)$ , which is a map  $C_n(A \cap B) \rightarrow \mathbb{Z}$ , to a map  $\psi: C_n(A) \rightarrow \mathbb{Z}$ , i.e. a cochain  $\psi \in C^n(A)$ .

Correction 30 April

For each  $x \in C_{n+1}(X)$ , choose  $y \in C_{n+1}(A)$ ,  $z \in C_{n+1}(B)$  such that  $x = (y+z)$  is a boundary. Then send  $[\varphi]$  to the cohomology class in  $H^{n+1}(X)$  that sends each  $x$  to  $\psi(dy)$ .

Thm 14 (Good Pairs)  $A \subseteq X$  non-empty closed,  $A$  a deformation retract of an open neighbourhood of  $A$  in  $X \Rightarrow$  the projection  $(X, A) \rightarrow (X/A, \{*\})$  induces an iso

$$\underbrace{H^n(X/A, \{*\})}_{\cong \tilde{H}^n(X/A)} \longrightarrow H^n(X, A)$$

Def For  $X \neq \emptyset$ , the  $n$ -th reduced cohomology group  $\tilde{H}^n(X; M)$  is the  $n$ -th cohomology group of the augmented cochain complex

$$0 \rightarrow M \xrightarrow{\varepsilon} C^0(X; M) \longrightarrow C^1(X; M) \longrightarrow \dots$$

with  $\varepsilon(m)(\sigma) = m$  for all  $\sigma: \Delta^0 \rightarrow X$ .

Prop 15  $H^n(X; M) \cong \tilde{H}^n(X; M)$  for  $n \geq 1$ ,

$$H^0(X; M) \cong \tilde{H}^0(X; M) \oplus M$$



Ex 16  $\tilde{H}^n(S^k) \cong \mathbb{Z}^{s(n,k)}$

$k=0$ :  $\checkmark$ . Assume now  $k \geq 1$ .

1st Proof  $C_{\bullet}^{\bullet}(S^k) \cong \text{Hom}(C_{\bullet}^{cw}(S^k), \mathbb{Z}) \cong C_{\bullet}^{cw}(S^k)$

2nd Proof  $H_{\bullet}(S^k)$  free  $\xrightarrow{uct} H^n(S^k) \cong H_n(S^k)$

3rd Proof  $A = S^k \setminus \{e_1\}, B = S^k \setminus \{-e_1\} \Rightarrow A, B$  contractible

$\Rightarrow$  Mayer-Vietoris gives us  $H^i(\underbrace{A \cap B}_{\cong S^{k-1}}) \longrightarrow H^{i+1}(S^k)$

Proceed by induction.

4th Proof  $H^i(S^k)$

$$\begin{array}{ccc} \text{iso} \downarrow & \text{LES of Pair } (D^{k+1}, S^k) & \\ H^{i+1}(D^{k+1}, S^k) & & \\ \text{iso} \downarrow & \text{due to good pair} & \\ H^{i+1}(S^{k+1}) & & \end{array}$$

Prop 17 Let  $n \geq 1$ . If  $f: S^n \rightarrow S^n$  has degree  $k \in \mathbb{Z}$ , then

$f^*: H^n(S^n) \rightarrow H^n(S^n)$  is multiplication by  $k$ .

Reminder "f has degree k" is by def equivalent to:

$f_*: H_n(S^n) \rightarrow H_n(S^n)$  is multiplication by  $k$

1st Proof

$$\cdots \xrightarrow{\circ} C_n^{cw}(S^n) \xrightarrow{\circ} \cdots$$

$$\downarrow f_c = \text{mult by } k$$

$$\cdots \xrightarrow{\circ} C_n^{cw}(S^n) \xrightarrow{\circ} \cdots$$

$$\xrightarrow[\text{Hom}(\cdot, \mathbb{Z})]{\text{apply functor}}$$

$$\cdots \xleftarrow{\circ} C_n^{cw}(S^n) \xleftarrow{\circ} \cdots$$

$$\downarrow f^c = \text{Hom}(f_c, \mathbb{Z}) = \text{mult by } k$$

$$\cdots \xleftarrow{\circ} C_n^{cw}(S^n) \xleftarrow{\circ} \cdots$$

2nd Proof Use naturality of UCT. (Skipped in lecture)

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$\text{Ext}(H_{n-1}(S^n), \mathbb{Z}) \cong 0$  since  $H_{n-1}(S^n)$  is free (namely, it is 0 (if  $n \geq 2$ ) or  $\mathbb{Z}$  (if  $n=1$ )). So we have an iso

$$\text{ev}: H^n(S^n) \longrightarrow \text{Hom}(H_n(S^n), \mathbb{Z})$$

It is natural, so the following commutes:

$$\begin{array}{ccc} H^n(S^n) & \xrightarrow[\text{iso}]{\text{ev}} & \text{Hom}(H_n(S^n), \mathbb{Z}) \\ \downarrow f^* & & \downarrow \text{Hom}(f_*, \mathbb{Z}) = \text{mult by } k \\ H^n(S^n) & \xrightarrow[\text{iso}]{\text{ev}} & \text{Hom}(H_n(S^n), \mathbb{Z}) \end{array} \quad \square$$

## ⑥ The cup product

**Reminder about simplexes** If  $v_0, \dots, v_n \in \mathbb{R}^k$  s.t.  $v_1 - v_0, \dots, v_n - v_0$  are lin indep., then the convex hull of  $\{v_0, \dots, v_n\}$ , ie

$$\left\{ \sum_{i=0}^n \lambda_i v_i \mid \sum_{i=0}^n \lambda_i = 1, (\lambda_0, \dots, \lambda_n) \in [0, 1]^{n+1} \right\} \subseteq \mathbb{R}^k$$

together with the tuple  $(v_0, \dots, v_n)$ , is called an **n-simplex**, denoted  $[v_0, \dots, v_n]$ . Every pair of n-simplexes  $[v_0, \dots, v_n]$ ,  $[v'_0, \dots, v'_n]$  is naturally homeomorphic via  $\sum \lambda_i v_i \mapsto \sum \lambda_i v'_i$ .

The **standard n-simplex** is  $\Delta^n := [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$ .

A **singular n-simplex** of a top. space  $X$  is a cont. map  $\sigma: \Delta^n \rightarrow X$ .

They form the basis of  $C_n(X)$ . The **boundary operator**

$d: C_n(X) \rightarrow C_{n-1}(X)$  is given by  $d(\sigma) = \sum_{i=0}^n \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}$ .  
means  $e_i$  is left out

(where we implicitly identify the non-standard simplex  $[e_0, \dots, \hat{e}_i, \dots, e_n]$  with  $\Delta^{n-1}$  via the natural homeo).

Throughout, let  $R$  be a commutative unital ring.

**Def**  $X$  top space,  $\varphi \in C^n(X; R)$ ,  $\psi \in C^k(X; R)$ .

Let the **cup-product**  $\varphi \smile \psi \in C^{n+k}(X; R)$

$\smile$  (smile, not \cup, in LaTeX)

be given sending singular simplexes  $\sigma: \Delta^{n+k} = [e_0, \dots, e_{n+k}] \rightarrow X$  to

$$(\varphi \smile \psi)(\sigma) = \underbrace{\varphi(\sigma|_{[e_0, \dots, e_n]})}_{\substack{\text{"front face" of } \sigma}} \cdot \underbrace{\psi(\sigma|_{[e_n, \dots, e_{n+k}]})}_{\substack{\text{"back face" of } \sigma}}$$

multiplication in  $R$

**Prop 1** (1)  $\cup: C^n(X; R) \times C^k(X; R) \longrightarrow C^{n+k}(X; R)$

$\cup$  is  $R$ -bilinear. (uses distributivity & associativity of  $R$ )

(2)  $\cup$  is associative:  $(\varphi \cup \psi) \cup \eta = \varphi \cup (\psi \cup \eta)$   
(uses associativity of  $R$ )

(3) Let  $\varepsilon \in C^0(X; R)$ ,  $\varepsilon(\sigma) = 1 \in R$  for all  $\sigma$ . Then  
 $\varphi \cup \varepsilon = \varepsilon \cup \varphi = \varphi$ . (uses unit of  $R$ )

**Proof** Exercise.

**Remark 2**  $\cup$  makes  $C^\bullet(X; R) = \bigoplus_{n=0}^{\infty} C^n(X; R)$  into a

(generally non-commutative) unital  $R$ -algebra (by Prop 1).

Moreover,  $C^\bullet(X; R)$  is graded:

a **grading** on an  $R$ -algebra  $S$  is a decomposition

$S = \bigoplus_{n \in \mathbb{Z}} S_n$  as an  $R$ -module, such that  $S_n S_k \subseteq S_{n+k}$ .

We write **deg**  $x = n$  for  $x \in S_n, x \neq 0$ . **deg** is not defined if  $x \notin S_n \forall n$ .

**Example 3**  $C^\bullet(\emptyset; R) =$  the zero ring

$C^\bullet(\{*\}; R)$ : For all  $n \geq 0$ ,  $C_n(\{*\})$  is generated by the constant  $\sigma_n: \Delta^n \rightarrow \{*\}$ , and  $C^n(\{*\}; R)$  by  $\varphi_n: \sigma_n \mapsto 1$ .

Check  $\varphi_n \cup \varphi_k = \varphi_{n+k}$ . So we have an isomorphism of graded  $R$ -algebras:  $C^\bullet(\{*\}; R) \longrightarrow R[x]$ ,  $\varphi_n \mapsto x^n$ .

Here, **deg** on  $R[x]$  is different from the usual **deg** of polynomials:  
 $\deg(rx^n) = n$ , **deg** not defined for non-monomials.

**Prop 4** (Graded Leibniz rule). For  $\varphi \in C^n(X; R)$ ,  $\psi \in C^k(X; R)$ :

$$d(\varphi \cup \psi) = (d\varphi) \cup \psi + (-1)^n \varphi \cup d\psi$$

 Koszul sign rule:

"when  $d$  jumps over something of degree  $k$ ,  $(-1)^k$  appears"

Calculate:

Proof

$$((d\varphi) \cup \varphi)(\sigma: [e_0, \dots, e_{m+k+1}] \rightarrow X)$$

$$= (d\varphi)(\sigma|_{[e_0, \dots, e_{m+1}]}) \cdot \varphi(\sigma|_{[e_{m+1}, \dots, e_{m+k+1}]})$$

$$= \varphi(d\sigma|_{\dots}) \cdot \varphi(\dots)$$

$$= \varphi\left(\sum_{i=0}^{m+1} (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{m+1}]}\right) \cdot \varphi(\dots)$$

$$= \sum_{i=0}^{m+1} (-1)^i \varphi(\sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{m+1}]}) \varphi(\sigma|_{[e_{m+1}, \dots, e_{m+k+1}]})$$

and:

$$(\varphi \cup d\varphi)(\sigma) =$$

$$= \sum_{j=0}^{k+1} (-1)^j \varphi(\sigma|_{[e_0, \dots, e_n]}) \varphi(\sigma|_{[e_n, \dots, \hat{e}_{n+j}, \dots, e_{m+k+1}]})$$

Now plug this into:

$$((d\varphi) \cup \varphi)(\sigma) + (-1)^m (\varphi \cup d\varphi)(\sigma)$$

Notice the last summand ( $i=n+1$ ) cancels the first ( $j=0$ )!

$$= \sum_{i=0}^n (-1)^i \varphi(\sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{n+1}]}) \varphi(\sigma|_{[e_{n+1}, \dots, e_{m+k+1}]})$$

$$+ \sum_{m=n+1}^{n+k+1} (-1)^m \varphi(\sigma|_{[e_0, \dots, e_n]}) \varphi(\sigma|_{[e_n, \dots, \hat{e}_m, \dots, e_{m+k+1}]})$$

index shift  $m = j + n$ 

$$= (d(\varphi \cup \varphi))(\sigma)$$

□

Prop 5

$$(1) \text{ cocycle} \cup \text{cocycle} = \text{cocycle}$$

$$(2) \text{ coboundary} \cup \text{cocycle} = \text{coboundary} \quad \text{and} \\ \text{cocycle} \cup \text{coboundary} = \text{---} \text{---} \text{---}$$

$$(3) \text{ For } [\varphi] \in H^n(X; \mathbb{R}), [\psi] \in H^k(X; \mathbb{R}), \\ [\varphi] \cup [\psi] := [\varphi \cup \psi] \in H^{n+k}(X; \mathbb{R}) \text{ is well-def}$$

$$(4) \cup \text{ makes } H^\bullet(X; \mathbb{R}) := \bigoplus_{i=0}^{\infty} H^i(X; \mathbb{R}) \text{ into a} \\ \text{graded } \mathbb{R}\text{-algebra.}$$

Proof

$$(1) \text{ If } d\varphi = d\psi = 0 \Rightarrow d(\varphi \cup \psi) = (d\varphi) \cup \psi \pm \varphi \cup d\psi = 0.$$

$$(2) \text{ If } \varphi = d\eta \text{ and } d\psi = 0 \Rightarrow \varphi \cup \psi = (d\eta) \cup \psi = d(\eta \cup \psi).$$

$$(3) \varphi \cup \psi \text{ is a cocycle by (1).}$$

$$\text{If } \varphi' = \varphi + d\eta, \psi' = \psi + d\xi, \text{ then}$$

$$[\varphi' \cup \psi'] = [\varphi \cup \psi] + \underbrace{[\varphi \cup d\xi]}_{=0} + \underbrace{[d\eta \cup \psi]}_{=0} \quad \text{by (2)}$$

$$(4) \text{ Follows from Prop 1}$$

□

**Example 6** If  $l \geq 1$ , then  $H^*(S^l; \mathbb{R}) \cong \mathbb{R}[x]/(x^2)$  with  $\deg x = l$  ( $x^2 = 0$  since there is no non-trivial cohomology class of  $\deg 2l$ ).

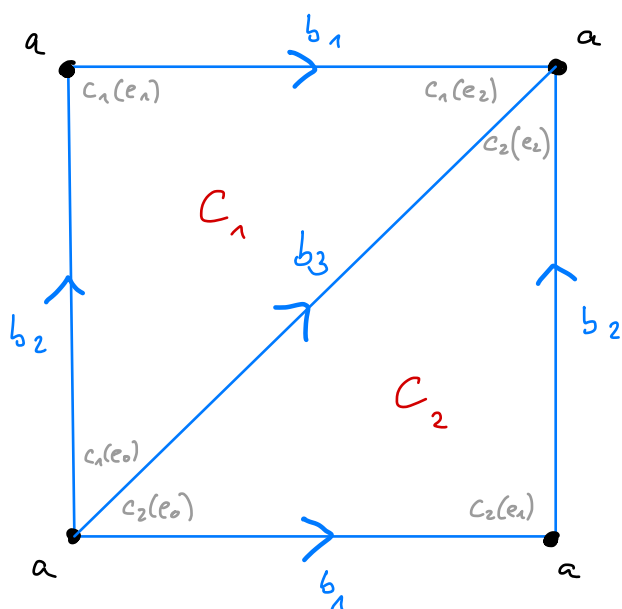
**Def** For a  $\Delta$ -complex  $X$ , define  $\smile$  in the same way as before on the simplicial cochain complex  $C_\Delta^\bullet(X; \mathbb{R}) = \text{Hom}(C_\bullet^\Delta(X), \mathbb{R})$ , and on its cohomology  $H_\Delta^\bullet(X; \mathbb{R})$ .

**Prop 7** The chain homotopy equivalence  $C_\bullet^\Delta(X) \rightarrow C_\bullet(X)$ , sending simplex to simplex, induces a chain homotopy equivalence  $C^\bullet(X) \rightarrow C_\Delta^\bullet(X)$  that preserves the cup product. Thm 2.27 in Hatcher

**Proof** Immediate from def □

**Example 8**  $X = S^1 \times S^1$ . Know  $H^0(X) \cong \mathbb{Z}$ ,  $H^1(X) \cong \mathbb{Z}^2$ ,  $H^2(X) \cong \mathbb{Z}$ . So  $\smile$  may be interesting on  $H^1(X)$ .

Put a  $\Delta$ -complex-structure on  $X$ :



$$a \in C_0^\Delta(X), \quad b_1, b_2, b_3 \in C_1^\Delta(X)$$

$$c_1, c_2 \in C_2^\Delta(X) \Rightarrow$$

$$db_i = 0,$$

$$dc_1 = dc_2 = b_1 - b_3 + b_2$$

One computes that:

$$H_0^\Delta(X; \mathbb{Z}) \text{ has basis } [a]$$

$$H_1^\Delta(X; \mathbb{Z}) \text{ --- } [b_1], [b_2]$$

$$H_2^\Delta(X; \mathbb{Z}) \text{ --- } [c_1 - c_2]$$

Since  $H_{\bullet}^{\Delta}(X; \mathbb{Z})$  is torsion-free, the UCT implies

$H_{\Delta}^{\bullet}(X; \mathbb{Z}) \cong \text{Hom}(H_{\bullet}^{\Delta}(X; \mathbb{Z}))$ . So the dual basis of the basis  $[a], [b_1], [b_2], [c_1 - c_2]$  is a basis for  $H_{\Delta}^{\bullet}(X; \mathbb{Z})$ :

$$\begin{array}{ccccc} [\varphi], [\psi^1], [\psi^2], [\eta] \\ \text{deg} & 0 & 1 & 1 & 2 \end{array}$$

with  $\varphi(a) = 1$ ,  $\psi^i(b_j) = \delta_{ij}$ ,  $\eta(c_1 - c_2) = 1$ .

Let's calculate  $[\psi^1] \cup [\psi^2]$ ! Since  $[\psi^1] \cup [\psi^2] \in H^2(X; \mathbb{Z})$   
 $\Rightarrow [\psi^1] \cup [\psi^2] = \lambda [\eta]$  for some  $\lambda \in \mathbb{Z}$ .

Evaluate both sides on  $[c_1 - c_2]$ :

$$\begin{aligned} \lambda &= \text{ev}([\psi^1] \cup [\psi^2])([c_1 - c_2]) \\ &= \text{ev}([\psi^1 \cup \psi^2])([c_1 - c_2]) && \text{by def of } \cup \text{ on cohomology} \\ &= (\psi^1 \cup \psi^2)(c_1 - c_2) && \text{by def of ev} \\ &= (\psi^1 \cup \psi^2)(c_1) - (\psi^1 \cup \psi^2)(c_2) && \text{by linearity} \\ &= \psi^1(c_1|_{[e_0, e_1]}) \psi^2(c_1|_{[e_1, e_2]}) - \psi^1(c_2|_{[e_0, e_1]}) \psi^2(c_2|_{[e_1, e_2]}) \\ & && \text{by def of } \cup \text{ on cochains} \\ &= \psi^1(b_2) \psi^2(b_1) - \psi^1(b_1) \psi^2(b_2) \\ &= -1 \end{aligned}$$

$$\Rightarrow [\psi^1] \cup [\psi^2] = -[\eta].$$

Similarly, one computes  $[\psi^2] \cup [\psi^1] = [\eta]$

$$\text{and } [\psi^i] \cup [\psi^i] = 0.$$

So  $H^{\bullet}(S^1 \times S^1; \mathbb{Z}) \cong \underbrace{\mathbb{Z}\langle x, y \rangle}_{\text{free algebra generated by } x, y} / (xy = -yx, x^2 = y^2 = 0)$



# Prop 9 (Naturality of $\cup$ )

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$f: X \rightarrow Y$  cont. map of top. spaces,  $[\varphi] \in H^n(Y; \mathbb{R}), [\psi] \in H^k(Y; \mathbb{R})$   
 $\Rightarrow f^*([\varphi] \cup [\psi]) = (f^*[\varphi]) \cup (f^*[\psi])$

## Proof (skipped in the lecture)

$$\begin{aligned} \text{For all } (n+k)\text{-simplex } \sigma: (f^c(\varphi \cup \psi))(\sigma) &= \varphi \cup \psi(f \circ \sigma) \\ &= \varphi(f \circ \sigma|_{[e_0, \dots, e_n]}) \psi(f \circ \sigma|_{[e_n, \dots, e_{n+k}]}) \\ &= f^c \varphi(\sigma|_{\dots}) \cdot f^c \psi(\sigma|_{\dots}) = ((f^c \varphi) \cup (f^c \psi))(\sigma). \end{aligned}$$

$$\begin{aligned} \text{Now } f^*([\varphi] \cup [\psi]) &= f^*([\varphi \cup \psi]) = [f^c(\varphi \cup \psi)] \\ &= [(f^c \varphi) \cup (f^c \psi)] = [f^c \varphi] \cup [f^c \psi] = \\ &= f^*([\varphi]) \cup f^*([\psi]) \quad \square \end{aligned}$$

In other words:  $f^*$  is a homomorphism of graded  $\mathbb{R}$ -algebras!

## Prop 10 $X, Y$ top spaces $\Rightarrow$ We have graded $\mathbb{R}$ -algebra isom

$$(1) H^*(X \sqcup Y; \mathbb{R}) \xrightarrow{\begin{pmatrix} \text{incl}^* \\ \text{incl}^* \end{pmatrix}} H^*(X; \mathbb{R}) \times H^*(Y; \mathbb{R})$$

$$(2) H^*(X \vee Y; \mathbb{R}) \xrightarrow{\begin{pmatrix} \text{incl}^* \\ \text{incl}^* \end{pmatrix}} \text{Subalgebra of } H^*(X; \mathbb{R}) \times H^*(Y; \mathbb{R})$$

containing in deg 0 only those  $(\varphi, \psi)$  with  $\varphi(x_0) = \psi(y_0)$

wedge product  $X \sqcup Y / \{x_0\} \sim \{y_0\}$  for some  $x_0 \in X, y_0 \in Y$  that are deformation retracts of neighbourhoods  $N_x, N_y$ .

Proof (1) We know  $\begin{pmatrix} \text{incl}^* \\ \text{incl}^* \end{pmatrix}$  is an  $\mathbb{R}$ -module isom. (eg use MV).

It's an algebra homom by Prop 9.

(2) Mayer-Vietoris gives isom for  $n \geq 1$ , and a SES

$$0 \rightarrow H^*(X \vee Y; \mathbb{R}) \rightarrow H^*(\underbrace{X \cup N_y}_{\simeq X}) \oplus H^*(\underbrace{Y \cup N_x}_{\simeq Y}) \rightarrow H^*(\underbrace{N_x \cap N_y}_{\simeq \{x\}}) \rightarrow 0$$

the kernel is the desired subalgebra  $\square$

**Example 11**  $H^*(S^1 \vee S^1 \vee S^2) \cong$

$$\mathbb{Z} \langle x_1, x_2, x_3 \rangle / (x_i x_j = 0 \text{ for all } i, j)$$

$$\deg x_1 = \deg x_2 = 1, \deg x_3 = 2$$

This is not isomorphic to the ring  $H^*(S^1 \times S^1)$ , which contains elements of degree 1 with non-zero product.

$$\Rightarrow S^1 \vee S^1 \vee S^2 \not\cong S^1 \times S^1$$

**Theorem 13**  $X$  top. space,  $A \subseteq X$ ,  $\varphi \in H^n(X, A; \mathbb{R})$ ,

$\psi \in H^k(X, A; \mathbb{R})$ . Then

$$\varphi \smile \psi = (-1)^{nk} \psi \smile \varphi$$

**Proof:** next lecture.

This property of the graded  $\mathbb{R}$ -alg.  $H^*(X, A; \mathbb{R})$  is called **graded commutative**.

**Theorem 13**  $X$  top. space,  $[\varphi] \in H^n(X; \mathbb{R})$ ,

$[\psi] \in H^k(X; \mathbb{R})$ . Then

Hatcher Thm 3.11, p. 210

$$[\varphi] \smile [\psi] = (-1)^{nk} [\psi] \smile [\varphi].$$

**Proof** For  $\sigma: \Delta^n \rightarrow X$ , let  $\bar{\sigma}: \Delta^n \rightarrow X$

be  $\bar{\sigma} = \sigma \circ (\text{natural homeo } [e_0, \dots, e_n] \rightarrow [e_n, e_{n-1}, \dots, e_1, e_0])$ ,

i.e.  $\bar{\sigma}(e_i) = \sigma(e_{n-i})$ . Let  $p: C_\bullet(X) \rightarrow C_\bullet(X)$ ,  $\sigma \mapsto (-1)^{\varepsilon_n} \bar{\sigma}$ ,

where  $\varepsilon_n = \frac{(n+1)n}{2}$ .

Claim 1:  $p$  is a chain map.

Claim 2:  $p \simeq \text{id}_{C_\bullet(X)}$

Pf that Claim 1 & 2  $\Rightarrow$  Thm:

$$(p^*(\varphi \smile \psi))(\sigma) = (-1)^{\varepsilon_{n+k}} \varphi(\sigma|_{[e_{n+k}, \dots, e_k]}) \psi(\sigma|_{[e_k, \dots, e_0]})$$

$$((p^*\psi) \smile (p^*\varphi))(\sigma) = (-1)^{\varepsilon_n + \varepsilon_k} \psi(\sigma|_{[e_k, \dots, e_0]}) \varphi(\sigma|_{[e_{n+k}, \dots, e_k]})$$

$$\Rightarrow [\varphi] \smile [\psi] = [\varphi \smile \psi] = [p^*(\varphi \smile \psi)]$$

$$= (-1)^{\varepsilon_{n+k} + \varepsilon_n + \varepsilon_k} [(p^*\psi) \smile (p^*\varphi)] = (-1)^{nk} [p^*\psi] \smile [p^*\varphi]$$

$$= (-1)^{nk} [\psi] \smile [\varphi]. \quad \text{Check that } \varepsilon_{n+k} + \varepsilon_n + \varepsilon_k \equiv nk \pmod{2} \checkmark$$

Pf of Claim 1:  $p \circ \sigma = p\left(\sum_{i=0}^n (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}\right)$

$$= \sum_{i=0}^n (-1)^{i + \varepsilon_{n-1}} \sigma|_{[e_n, \dots, \hat{e}_i, \dots, e_0]}$$

$$d p \sigma = \sum_{j=0}^n (-1)^{j + \varepsilon_n} \sigma|_{[e_n, \dots, \hat{e}_{n-j}, \dots, e_0]} \quad n-j=i$$

$$= \sum_{i=0}^n (-1)^{n-i + \varepsilon_n} \sigma|_{[e_n, \dots, \hat{e}_i, \dots, e_0]}$$

Check:  $\varepsilon_{n-1} \equiv n + \varepsilon_n \pmod{2} \Leftrightarrow n + \frac{n(n-1)}{2} \equiv \frac{n(n+1)}{2} \checkmark$

Pf of Claim 2: Need homotopy  $s: C_n(x) \rightarrow C_{n+1}(x)$  with

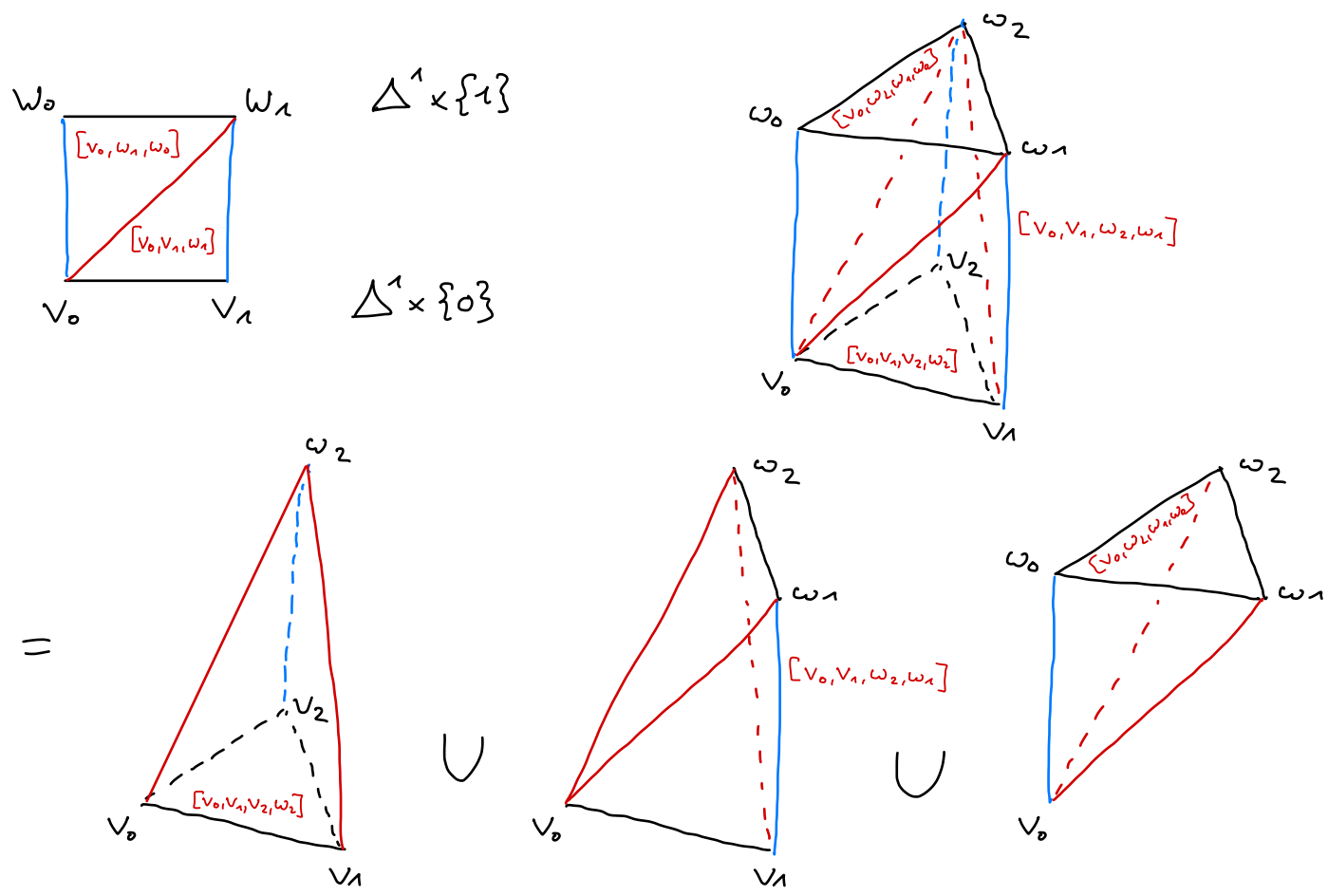
$$d_{n+1}s_n + s_{n-1}d_n = p_n - \text{id}_{C_n}. \quad (*)$$

Construction of  $s$  is inspired by the prism operator:

cut the prism  $\Delta^n \times [0, 1] \subseteq \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$

into  $n+1$  many  $(n+1)$ -simplices.

Let  $v_i = (e_i, 0)$  and  $w_i = (e_i, 1)$  for  $i = 0, \dots, n$ .



Let  $\pi: \Delta^n \times [0, 1] \rightarrow \Delta^n$  be the projection, so that  $\pi(w_i) = \pi(v_i) = e_i$ .

Define

$$S_n(\sigma) := \sum_{i=0}^n (-1)^{i+\varepsilon_{n-i}} \sigma \circ \pi([v_0, \dots, v_i, w_n, \dots, w_i])$$

Let us check by calculation that  $(*)$  holds.

$$d_{n+1}(s_n(\sigma)) = \sum_{0 \leq j \leq i \leq n}^{(1)} (-1)^{i + \varepsilon_{n-i} + j} \sigma \circ \pi([v_0, \dots, \hat{v}_j, \dots, v_i, \omega_n, \dots, \omega_i])$$

$$+ \sum_{0 \leq i \leq j \leq n}^{(2)} (-1)^{\varepsilon_{n-i} + n + j + 1} \sigma \circ \pi([v_0, \dots, v_i, \omega_n, \dots, \hat{\omega}_j, \dots, \omega_i])$$

index  $n-j+i+1$

Consider the summands with  $i=j$ :

$$(-1)^{\varepsilon_n} \sigma \circ \pi([\omega_n, \dots, \omega_0]) +$$

$$+ \sum_{i=1}^{n+1} (-1)^{\varepsilon_{n-i}} \sigma \circ \pi([v_0, \dots, v_{i-1}, \omega_n, \dots, \omega_i])$$

$$+ \sum_{k=0}^n (-1)^{\varepsilon_{n-k} + n + k + 1} \sigma \circ \pi([v_0, \dots, v_k, \omega_n, \dots, \omega_{k+1}])$$

$$+ (-1)^{\varepsilon_0} \sigma \circ \pi([v_0, \dots, v_n])$$

these cancel:  
index shift  $k=i-1$ , check  
 $\varepsilon_{n-i} \neq \varepsilon_{n-i+1} + n + i$  (2)

$$= (-1)^{\varepsilon_n} \bar{\sigma} + \sigma = \rho\sigma - \sigma$$

So, to prove (\*), one has to check that the summands with  $i \neq j$  equal  $-s_{n-1}(d_n(\sigma))$

$$= -s_{n-1} \left( \sum_{j=0}^n (-1)^j \sigma([v_0, \dots, \hat{v}_j, \dots, v_n]) \right)$$

$$= \sum_{0 \leq j \leq k \leq n} (-1)^{1+j+k+\varepsilon_{n-k-1}} \sigma \circ \pi([v_0, \dots, \hat{v}_j, \dots, v_{k+1}, \omega_n, \dots, \omega_{k+1}])$$

index shift:  $k=i-1$ . check  $i + \varepsilon_{n-i} + j \equiv 1 + j + i - 1 + \varepsilon_{n-i}$   
 $\Rightarrow$  equals summands of (1) with  $j < i$

$$+ \sum_{0 \leq i < j \leq n} (-1)^{1+j+i+\varepsilon_{n-i-1}} \sigma \circ \pi([v_0, \dots, v_i, \omega_n, \dots, \hat{\omega}_j, \dots, \omega_i])$$

check:  $\varepsilon_{n-i} + n + j + 1 \equiv 1 + j + i + \varepsilon_{n-i-1}$

$\Rightarrow$  equals summands of (2) with  $i < j$

□

**Remark 14** We'll prove later that:

$$H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[x] / (x^{n+1}) \quad \text{with } \deg x = 2$$

(commutative since  $H^k(\mathbb{C}P^n) = 0$  for odd  $k$ )

$$H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x] / (x^{n+1}) \quad \text{with } \deg x = 1$$

(commutative because of  $\mathbb{Z}/2$  coefficients)

$$H^\bullet((S^1)^{\times n}) \cong \mathbb{Z}\langle x_1, \dots, x_n \rangle / (x_i x_j + x_j x_i, x_i^2)$$

with  $\deg x_i = 1$

(not commutative, but graded commutative)

**Reminder from Alg Top 1**  $X$  top. space,  $A, B \subseteq X$ .

$C_n(A+B) \subseteq C_n(A \cup B)$  is generated by  $C_n(A) \cup C_n(B) \subseteq C_n(A \cup B)$ .

$C_n(A+B)$  is a chain complex, and  $C_\bullet(A+B) \xrightarrow{i} C_\bullet(A \cup B)$

is a homotopy equivalence (proved by barycentric subdivision).

**Lemma 14** There is a (natural) iso

$$H^n(X, A \cup B; \mathbb{R}) \xrightarrow{j} H^n(X, A+B; \mathbb{R}) \text{ induced by } i.$$

**Proof** (skipped in lecture)

$$\begin{array}{ccccccc} 0 & \rightarrow & C_n(A+B) & \rightarrow & C_n(X) & \rightarrow & C_n(X, A+B) \rightarrow 0 \\ & & \downarrow i & & \downarrow \text{id}_X & & \downarrow \\ 0 & \rightarrow & C_n(A \cup B) & \rightarrow & C_n(X) & \rightarrow & C_n(X, A \cup B) \rightarrow 0 \end{array}$$

commutes, has split exact rows. Apply  $\text{Hom}(-, \mathbb{R})$  and take the natural LES in cohomology:

$$\begin{array}{ccccccc} \dots & \leftarrow & H^n(A+B; \mathbb{R}) & \leftarrow & H^n(X; \mathbb{R}) & \leftarrow & H^n(X, A+B; \mathbb{R}) \leftarrow \dots \\ & & \uparrow i^* & & \uparrow \text{id} & & \uparrow j \\ \dots & \leftarrow & H^n(A \cup B; \mathbb{R}) & \leftarrow & H^n(X; \mathbb{R}) & \leftarrow & H^n(X, A \cup B; \mathbb{R}) \leftarrow \dots \end{array}$$

$j$  is an iso by the five lemma.

□

Def Let  $X$  be a top. space and  $A, B \subseteq X$ . Let the relative cup product

$$\cup: H^m(X, A; \mathbb{R}) \times H^k(X, B; \mathbb{R}) \rightarrow H^{m+k}(X, A \cup B; \mathbb{R})$$

be the postcomposition with  $j^{-1}$  of the bilinear map on cohomology induced by

$$\cup: C^m(X, A; \mathbb{R}) \times C^k(X, B; \mathbb{R}) \rightarrow C^{m+k}(X, A+B; \mathbb{R})$$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[e_0, \dots, e_n]}) \psi(\sigma|_{[e_n, \dots, e_{n+k}]})$$

$\uparrow$   
 $\text{im } \sigma \subseteq A \text{ or } \text{im } \sigma \subseteq B$

$\underbrace{\quad}_{0 \text{ if } \text{im } \sigma \subseteq A} \quad \underbrace{\quad}_{0 \text{ if } \text{im } \sigma \subseteq B}$

I Motivation

**Def (Poincaré algebra)** A connected ( $\Leftrightarrow A^0 = \mathbb{k}$ ) gr.  $A = \bigoplus_{i=0}^{\infty} A^i$  over a field  $\mathbb{k}$  is called a **Poincaré algebra** of formal dimension  $n$  if:

- (i)  $A^j = 0$  for  $j > n$ .
- (ii)  $A^n \cong \mathbb{k}$
- (iii) the bilinear pairing  $A^i \otimes A^{n-i} \longrightarrow A^n \cong \mathbb{k}$  is non-degenerate  
 $\Leftrightarrow$  the map  $A^i \longrightarrow \text{Hom}_{\mathbb{k}}(A^{n-i}, \mathbb{k})$  is an isomorphism.

**Claim** Let  $M^n$  be a closed connected orientable manifold.  
 Then  $H^*(M; \mathbb{Q})$  is a Poincaré algebra of formal dimension  $n$ .

II Manifolds

**Def (Topological manifold)** A Hausdorff second countable topological space  $M$  is called a **topological manifold** (resp. **top. manifold with boundary**) of dimension  $n$  if each point  $x \in M$  has a neighborhood homeomorphic to an open subset of  $\mathbb{R}^n$  (resp. of  $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ ).

**Def (Boundary)** Let  $M$  be a manifold with boundary. The subset  $\partial M$  of points  $x \in M$  that do not have a neighborhood homeomorphic to an open subset of  $\mathbb{R}^n$  is called the boundary of  $M$ .

**Def (Closed manifold)** A compact manifold without boundary is called **closed**.

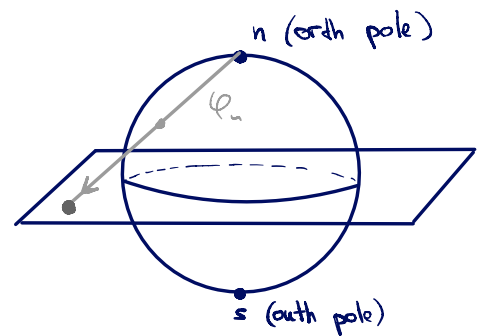
**Examples:** (i)  $\mathbb{R}^n$  any open subset of  $\mathbb{R}^n$ .

(ii)  $S^n := \{(x^1, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n (x^i)^2 = 1\}$

Two charts:  $\varphi_n: S^n \setminus \{n\} \longrightarrow \mathbb{R}^n$   
 $(x^1, \dots, x^n) \longmapsto \left( \frac{x^1}{1-x^n}, \dots, \frac{x^n}{1-x^n} \right)$

$\varphi_s: S^n \setminus \{s\} \longrightarrow \mathbb{R}^n$   
 $(x^1, \dots, x^n) \longmapsto \left( \frac{x^1}{1+x^n}, \dots, \frac{x^n}{1+x^n} \right)$

with transition maps:  $\varphi_s \circ \varphi_n^{-1}, \varphi_n \circ \varphi_s^{-1}: \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$   
 $(t^1, \dots, t^n) \longmapsto \left( \frac{t^1}{\|t\|^2}, \dots, \frac{t^n}{\|t\|^2} \right)$



(iii)  $n$ -dimensional torus  $T^n$ ;

(iv) real and complex projective spaces  $\mathbb{R}P^n$  &  $\mathbb{C}P^n$ .

... **with boundary:** (i)  $D^n$ ;

(ii) solid torus  $S^1 \times D^2$ .

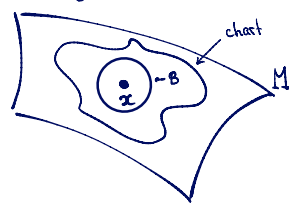
**Non-examples:** (i)  $\Delta$

(ii)  $\mathbb{R}P^{\infty} = \bigcup_{n=0}^{\infty} \mathbb{R}P^n$  &  $\mathbb{C}P^{\infty} = \bigcup_{n=0}^{\infty} \mathbb{C}P^n$

**Proposition 1** Let  $M^n$  be a topological manifold. Then for any  $x \in M$ :  $H_i(M, M \setminus \{x\}; \mathbb{R}) \cong \begin{cases} 0 & \text{if } i \neq n; \\ \mathbb{R} & \text{if } i = n. \end{cases}$

$\triangleright$  Let  $B$  be an open ball around  $x$  (sits inside of a neighborhood of  $x$  homeomorphic to a subset of  $\mathbb{R}^n$ ).

$\Rightarrow Z := M \setminus B$  is closed.





Def (Local (co)homology)  $H_i(M, M_{(x)}; \mathbb{R})$  local homology group  
 $H^i(M, M_{(x)}; \mathbb{R})$  local cohomology group.

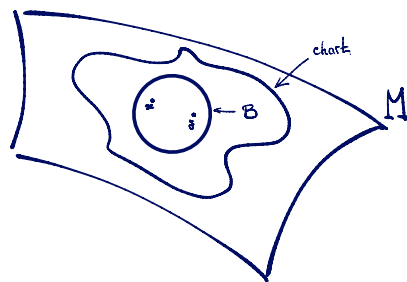
Def (Homology manifold) A Hausdorff second countable space is a **homology  $\mathbb{R}$ -manifold** of dimension  $n$  if for any  $x \in M$   $H_*(M, M \setminus \{x\}; \mathbb{R}) \cong \tilde{H}_*(S^n; \mathbb{R})$ .

Def (Local orientations) A local orientation  $\mu_x$  in  $x \in M$  is a generator of the local homology group  $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ .

Note that there are two choices of a generator in  $\mathbb{Z}$ .  
 $\Rightarrow$  At each point there are two possible orientations.

**Def (Orientation)** An **orientation** of an  $n$ -dimensional manifold is a choice of a local orientation  $\mu_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$  at every  $x \in M$ , st. it is locally consistent, i.e. if  $x, y \in M$  can be covered by a ball  $B$  within one chart. Then  $\mu_x$  and  $\mu_y$  map one to each other under the isomorphisms:

$$H_n(M, M|A; \mathbb{Z}) \xleftarrow{\cong} H_n(M, M|B; \mathbb{Z}) \xrightarrow{\cong} H_n(M, M|g; \mathbb{Z})$$



Def ((non-) Orientable manifold) A manifold is **orientable** if there exists an orientation on  $M$ .  
A manifold is **non-orientable** if it is not orientable.

Examples: (i)  $S^n$  is orientable.  
(ii) The Möbius band is non-orientable.

**Proposition 2** Let  $M$  be a closed connected manifold of dimension  $n$ .

- (i) The homomorphism  $H_n(M; \mathbb{F}_2) \rightarrow H_n(M, M \setminus \{x\}; \mathbb{F}_2)$  is an isomorphism for any  $x \in M$ .  
(ii) If  $M$  is orientable, then  $H_n(M; \mathbb{Z}) \rightarrow H_n(M, M \setminus \{x\}; \mathbb{Z})$  is an isomorphism for any  $x \in M$ .  
If  $M$  is non-orientable, then  $H_n(M; \mathbb{Z}) = 0$ .  
(iii)  $H_i(M; \mathbb{Z}) = 0$  for  $i > n$ .

Main Lemma 3 Let  $A \subseteq M$  be a compact subset of a manifold  $M$  of dimension  $n$ . (not necessary compact).

- (ii) For every locally consistent choice of orientations  $\mu_x, x \in A$ , exists a unique  $\mu_A \in H_n(M, M \setminus A; \mathbb{R})$  s.t. is  $\mu_x$  for all  $x \in A$ .

▷ STEP 1. If the assertion holds for compact  $A, B$  and  $A \cap B$ , then it holds for  $A \cup B$ .

Relative Mayer-Vietoris sequence:

$$H_{n+1}(M, M \setminus (A \cup B)) \rightarrow H_n(M, M \setminus (A \cup B)) \xrightarrow{\Phi} H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

For  $i > n$  we have  $H_i(M, M \setminus (A \cup B)) = H_i(M, M \setminus A) = H_i(M, M \setminus B) = 0 \Rightarrow H_i(M, M \setminus (A \cup B))$  is locked between two zeros  $\Rightarrow$  zero itself.

If  $\mu \in H_n(M, M \setminus (A \cup B))$  is s.t.  $\mu_x \in H_n(M, M \setminus \{x\})$  is zero for all  $x \in A \cup B \Rightarrow$  its images in  $H_n(M, M \setminus A)$  and  $H_n(M, M \setminus B)$  are zero by the assumption.  $\Rightarrow$  Since  $\Phi$  is injective,  $\mu = 0$ . (Proves (i)).

Let  $\mu_x, x \in A \cup B$  be a locally consistent choice of orientations  $\Rightarrow \exists! \mu_A \in H_n(M, M \setminus A), \mu_B \in H_n(M, M \setminus B)$   
 $\Psi(\mu_A, \mu_B) = \mu_A|_{A \cap B} - \mu_B|_{A \cap B} \in H_n(M, M \setminus (A \cap B))$ . its image is zero in  $H_n(M, M \setminus \{x\})$  for any  $x \in A \cap B$  since  $\Phi$  is injective  
 $\Rightarrow$  it is zero itself by assumption on  $A \cap B$ .  $\Rightarrow$  By exactness,  $(\mu_A, \mu_B)$  is the image of a unique element  $\mu_{A \cup B} \in H_n(M, M \setminus (A \cup B))$ . [1]

STEP 2 It is enough to prove the assertion for a compact subset of a single chart. (i.e. in  $\mathbb{R}^n$ )

Any compact subset  $A \subseteq M$  is a union of a finite number of compact subsets, s.t. each belongs to a chart  $\rightarrow$  We can apply induction and STEP 1.

If  $U$  is a chart, then  $H_i(M, M \setminus A) \cong H_i(U, U \setminus A)$  by excision. [2]

$\Rightarrow$  From now on we assume  $M = \mathbb{R}^n$ .

STEP 3 If  $A \subseteq \mathbb{R}^n$  is a finite simplicial complex, s.t. its simplices are linearly embedded, then the assertion follows by induction, and it is enough to prove for one simplex. The latter follows from the definition of local consistency. [3]

STEP 4.  $A \subseteq \mathbb{R}^n$  compact

$\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A)$  is represented by a relative cycle  $z$  and let  $C \subseteq \mathbb{R}^n \setminus A$  be a union of the images of the singular simplices of  $\partial z$ .

$A$  and  $C$  are compact  $\Rightarrow$  they have positive distance  $\delta > 0$  between them.

Cover  $A$  with a finite piecewise linear simplicial complex  $K$  with  $K \cap C = \emptyset$ :  
 (i) cover  $A$  by one big enough simplex;  
 (ii) take barycentric subdivision s.t. the diameter of a piece is less than  $\delta$ .  
 (iii) take simplices that intersect  $A$ .

The same chain  $z$  represents a class  $\alpha_K \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$  that maps to  $\alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ .

By STEP 3,  $\alpha_K = 0$  for  $i > n \Rightarrow \alpha = 0$  and  $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) = 0$  for  $i > n$ .

Finally, assume  $i = n$ . If  $\alpha_{K,x} = 0 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$  for all  $x \in A$ , then it also holds for all  $x \in K$ .

Indeed, for any simplex  $\Delta \in K$  and any  $x \in \Delta$  the map  $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Delta) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$  is an iso.

STEP 3 now implies that  $\alpha_K = 0 \Rightarrow \alpha = 0$ , which concludes the proof of (i) and uniqueness part in (ii).

Existence: let  $\alpha_A \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$  be the image of  $\alpha_B \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ , where  $B$  is a big ball containing  $A$ .  
↑  
exists by definition of local consistency. □

Last time: Proof of

26 April

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**Lemma 3**  $M^n$  without boundary,  $A \subseteq M$  compact,  $R$  commutative unital ring.

(i)  $H_i(M, M \setminus A; R) = 0$  for  $i > n$ .

$\alpha \in H_n(M, M \setminus A; R)$  is zero  $\Leftrightarrow$

image of  $\alpha$  in  $H_n(M, M \setminus \{x\}; R)$  is zero for all  $x \in A$ .

(ii)  $\mu_x$  locally consistent choice of orientation for  $x \in A$

$\Rightarrow$  exists unique  $\mu_A \in H_n(M, M \setminus A; R)$  mapping to  $\mu_x$  for all  $x \in A$

Today:

**Prop 2**  $M^n$  closed ( $\Leftrightarrow$  compact, no boundary) connected.

(i)  $H_n(M; \mathbb{F}_2) \rightarrow H_n(M, M \setminus \{x\}; \mathbb{F}_2)$  iso for all  $x \in M$ .

(ii)  $M$  orientable  $\Rightarrow H_n(M; \mathbb{Z}) \rightarrow H_n(M, M \setminus \{x\}; \mathbb{Z})$  iso  $\forall x \in M$ .

$M$  non-orientable  $\Rightarrow H_n(M; \mathbb{Z}) = 0$ .

(iii)  $H_i(M; \mathbb{Z}) = 0$  for  $i > n$ .

Note that (iii) follows from Lemma 3 (i) with  $A = M$ . For (i) & (ii), we'll also use Lemma 3, but need some more tools.

For  $M^n$  without boundary, let

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$\tilde{M} := \{ \mu_x \mid x \in M \text{ and } \mu_x \in H_n(M, M \setminus \{x\}) \text{ a local orientation} \}$

Note  $p: \tilde{M} \rightarrow M$ ,  $\mu_x \mapsto x$  is a 2:1 surjection. For  $B \subseteq \text{chart} \subseteq M$  an open ball and a generator  $\mu_B \in H_n(M, M \setminus B)$ , let

$U_{(\mu_B)} := \{ \mu_x \in \tilde{M} \mid x \in B, \mu_x \text{ image of } \mu_B \text{ under} \\ H_n(M, M \setminus B) \rightarrow H_n(M, M \setminus \{x\}) \}$

**Exercise** The  $U_{(\mu_B)}$  form the base of a topology on  $\tilde{M}$ , st  $p$  is a 2:1 covering.

**Def**  $p: \tilde{M} \rightarrow M$  is called the **orientation covering** of  $M$ .

Each  $\mu_x \in \tilde{M}$  has a canonical orientation  $\tilde{\mu}_x \in H_n(\tilde{M}, \tilde{M} \setminus \mu_x)$  corresponding to  $\mu_x$  under the isos

$$\begin{aligned} H_n(\tilde{M}, \tilde{M} \setminus \mu_x) &\xleftarrow{\text{excision}} H_n(U_{(\mu_B)}, U_{(\mu_B)} \setminus \mu_x) \\ &\longrightarrow H_n(B, B \setminus x) \xrightarrow{\text{excision}} H_n(M, M \setminus x) \end{aligned}$$

These are locally consistent, so  $\tilde{M}$  has a canonical orientation.

**Prop 4** If  $M$  is connected, then:  $\tilde{M}$  non-connected  $\Leftrightarrow M$  orientable

**Proof**  $M$  has orientation  $\mu_x \Rightarrow \tilde{M} = \underbrace{\{\mu_x \mid x \in M\}}_{\text{open}} \sqcup \underbrace{\{-\mu_x \mid x \in M\}}_{\text{open}}$

If  $\tilde{M}$  has two components  $N_1, N_2$ , then they inherit an orientation from  $\tilde{M}$ . Check that  $p|_{N_i}: N_i \rightarrow M$  are coverings. Then, they must be one-sheeted coverings, i.e. homeomorphisms.  $\square$

**Example**  $\tilde{S}^2 \cong S^2 \sqcup S^2$ ,  $\widetilde{\mathbb{R}P^2} \cong S^2$ ,  $\widetilde{\text{Klein Bottle}} \cong S^1 \times S^1$

Note that  $S^3 \rightarrow \mathbb{R}P^3$  is an orientable double covering, but not the orientation covering, which is  $\mathbb{R}P^3 \sqcup \mathbb{R}P^3 \rightarrow \mathbb{R}P^3$  (since  $\mathbb{R}P^3$  is orientable).

**Def** A **section** of  $p$  is a cont. map  $s: M \rightarrow \tilde{M}$  with  $ps = \text{id}_M$ .

Note that a section of a covering map has a component of  $\tilde{M}$  as image

**Prop 5**  $\mu_x$  is an orientation  $\Leftrightarrow x \mapsto \mu_x$  is a section of  $p$

**Pf** Exercise  $\square$

**Def**  $R$  commutative unital ring,  $M^n$  without boundary.

**Local  $R$ -orientation**:  $\mu_x$  is a generator of  $H_n(M, M \setminus x; R)$

**$R$ -orientation**: locally consistent choice of local  $R$ -orientations.

**$M$   $R$ -orientable**:  $\Leftrightarrow$  There exists an  $R$ -orientation

**Example** Every  $M$  is  $\mathbb{F}_2$ -orientable, since there is precisely one local  $\mathbb{F}_2$ -orientation at every point.

**Def** Let  $M_R := \{\alpha_x \mid x \in M, \alpha_x \in H_n(M, M \setminus \{x\}; \mathbb{R})\}$ ,  
with similar topology as  $\tilde{M}$ .

Note  $p_R: M_R \rightarrow M$  is an  $|\mathbb{R}|$ -sheeted covering.

**Prop 6** Let  $M_\tau = \{\alpha_x \mid \alpha_x \text{ is the image of } \mu_x \otimes \tau \text{ under the iso } H_n(M, M \setminus x) \otimes \mathbb{R} \rightarrow H_n(M, M \setminus x; \mathbb{R}) \text{ for } \mu_x \text{ a generator of } H_n(M, M \setminus x)\}$

Then:  $M_\tau \subseteq M_R$  is open;  $M_\tau = M_{-\tau}$ ;

$M_\tau \cap M_S = \emptyset$  for  $\tau \neq \pm S$ ;

$M_\tau \cong M$  if  $\tau = -\tau$ , and  $M_\tau \cong \tilde{M}$  if  $\tau \neq -\tau$ .

**Pf:** Exercise □

**Prop 7**  $\mu_x$  is an  $\mathbb{R}$ -orientation  $\Leftrightarrow$

$x \mapsto \mu_x$  is a section of  $p_R$  with each  $\mu_x$  a generator of  $H_n(M, M \setminus x; \mathbb{R})$

**Pf** Exercise, similar to Prop 5. □

**Prop 8** If  $0 = 2$  in  $\mathbb{R} \Rightarrow$  all  $M^n$  are  $\mathbb{R}$ -orientable

If  $0 \neq 2$  in  $\mathbb{R} \Rightarrow M^n$  is  $\mathbb{R}$ -orientable iff  
it is  $\mathbb{Z}$ -orientable

**Proof**  $0 = 2 \Rightarrow M_1 \cong M \Rightarrow p_R$  has a section to  $M_1 \Rightarrow M$  is  $\mathbb{R}$ -orientable

Assume  $0 \neq 2$ . Generators of  $H_n(M, M \setminus x; \mathbb{R})$  are of the form  $\mu_x \otimes u$

for  $\mu_x$  a gen. of  $H_n(M, M \setminus x)$  and  $u \in \mathbb{R}$  a unit. Then  $u \neq -u$

$\Rightarrow M_u \cong \tilde{M} \Rightarrow p_R$  has a section to  $M_u$  iff  $\tilde{M} \rightarrow M$  has a section. □

**Proof of Prop 2 (i) and (ii)** Pointwise sum and pointwise  $R$ -multiplication turn  $\Gamma(M, M_R)$  into an  $R$ -module.

$$H_n(M; R) \longrightarrow \Gamma(M, M_R), \\ \alpha \mapsto (x \mapsto \text{image of } \alpha \text{ in } H_n(M, M \setminus x; R))$$

is a homomorphism. By Lemma 3, applied to  $A = M$ , it is an isomorphism! Indeed, Lemma 3 (i) yields injectivity. And Lemma 3 (ii) yields surjectivity (here, we need a slightly more general version of Lemma 3 (ii): namely, for every locally consistent choice  $\alpha_x \in H_n(M, M \setminus x; R)$ ,  $\exists! \mu_A \in H_n(M, M \setminus A; R)$  that maps to  $\alpha_x$  for all  $x$ . The proof is the same — we never use that  $\alpha_x$  generates).

$$M \text{ } R\text{-orientable} \Rightarrow \left\{ \begin{array}{ll} \tilde{M} = M \sqcup M & \text{if } 0 \neq 2 \\ M_r = M \text{ for all } r \in R & \text{if } 0 = 2 \end{array} \right\} \Rightarrow M_R \cong \bigsqcup_{r \in R} M \\ \Rightarrow \Gamma(M, M_R) \cong R \text{ (using connectedness of } M) \Rightarrow H_n(M; R) \cong R. \\ \text{So } H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2 \text{ for all } M \text{ (since all } M \text{ are } \mathbb{F}_2\text{-orientable),} \\ \text{and } H_n(M) \cong \mathbb{Z} \text{ for all orientable } M.$$

$$M \text{ non-orientable} \Rightarrow \tilde{M} \text{ is connected} \Rightarrow$$

$$M_{\mathbb{Z}} \cong \underbrace{M_0}_{\cong M} \sqcup \underbrace{M_1}_{\cong \tilde{M}} \sqcup \underbrace{M_2}_{\cong \tilde{M}} \dots$$

$$\text{So the only section of } p_{\mathbb{Z}} \text{ goes to } M_0 \Rightarrow \Gamma(M, M_{\mathbb{Z}}) \cong 0 \\ \Rightarrow H_n(M) \cong 0. \quad \square$$

**Corollary 9** (i) Let  $M$  be a closed  $\mathbb{R}$ -oriented  $n$ -manifold. Then there exists a unique class  $\mu \in H_n(M; \mathbb{R})$  st for all  $x \in M$ , the isom  $H_n(M, M \setminus \{x\}; \mathbb{R})$  sends  $\mu$  to the given local orientation.

(ii) If  $M$  is connected, then  $\mu$  generates  $H_n(M; \mathbb{R}) \cong \mathbb{R}$ .

**Proof** (i) directly from Lemma 3, (ii) similar to Prop 2.  $\square$

**Def** The class from Corollary 9 is called the **fundamental class** of  $M$ , written  $[M]_R \in H_n(M; \mathbb{R})$ .  
drop  $R$  from notation for  $R = \mathbb{Z}$ .

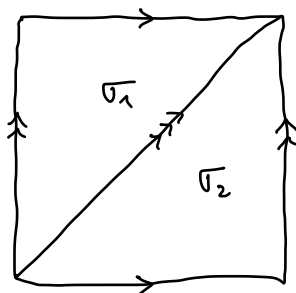
**Remark 10** If  $M^n$  is closed and has a  $\Delta$ -complex structure, then:

- (1) Every simplex of  $M$  is a subsimplex of an  $n$ -simplex.
- (2) Every  $(n-1)$ -simplex is a face of precisely two  $n$ -simplexes.
- (3)  $M$  has only finitely many  $n$ -simplexes  $\sigma_1, \dots, \sigma_k$ .

If  $M$  is oriented, then  $[M] = \left[ \sum_{i=1}^k \varepsilon_i \sigma_i \right]$  with  $\varepsilon_i = \pm 1$ .

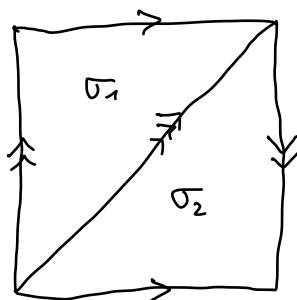
such that in  $\sum_{i=1}^k \varepsilon_i d\sigma_i$ , each  $(n-1)$ -simplex appears once with  $+$ , once with  $-$ . If  $M$  is not orientable, no such choice of  $\varepsilon_i$  exists. Over  $\mathbb{F}_2$ ,  $[M]_{\mathbb{F}_2} = \left[ \sum_{i=1}^k \sigma_i \right]$ .

For example:



Torus  $T$

$$[T] = \pm [\sigma_1 - \sigma_2]$$



Klein bottle  $K$

$$[K]_{\mathbb{F}_2} = [\sigma_1 + \sigma_2]$$

**Def**  $M^n, N^n$  compact, oriented, connected,  $f: M \rightarrow N$  continuous.

Then the **degree of  $f$**  is the unique integer  **$\deg f$**  st

$$f_*([M]) = \deg f \cdot [N] \in H_n(N).$$

For not necessarily orientable  $M, N$ , there is a unique  **$\deg_{\mathbb{F}_2} f \in \mathbb{F}_2$**  st

$$f_*([M]_{\mathbb{F}_2}) = \deg_{\mathbb{F}_2} f \cdot [N]_{\mathbb{F}_2} \in H_n(N; \mathbb{F}_2)$$

This extends our previous def of  $\deg$  for  $f: S^n \rightarrow S^n$ .

**Remark 11**  $\deg f \circ g = \deg f \cdot \deg g$  easily follows.

**Theorem 12 (Hopf 1927)**

$f, g: M^n \rightarrow S^n$  for  $M$  compact, connected, oriented. Then:

$$f \simeq g \iff \deg f = \deg g.$$

**Conjecture 13 (Hopf 1931)**

$f: M^n \rightarrow M^n$  for  $M$  compact, connected, oriented. Then:

$$f \simeq \text{id}_M \iff \deg f = 1$$

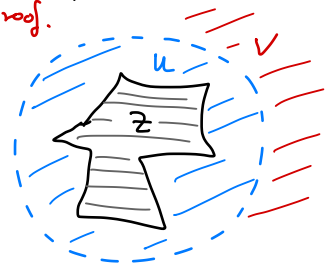


**Proposition 14**  $M^n$  non-compact and connected

$$\Rightarrow H_i(M; \mathbb{R}) = 0 \text{ for all } i \geq n.$$

**Proof** Let  $[z] \in H_i(M)$ . To show:  $[z] = 0$ . Pick  $U \subseteq M^n$  open st  
 $\text{im}(z) \subseteq U$  and  $\bar{U}$  compact. Let  $V = M \setminus \bar{U}$ .  
 ↑ Drop the  $\mathbb{R}$  from notation in this proof.

Consider the LES of  $(M, U \cup V, V)$ :



$$\begin{array}{ccccc} H_{i+1}(M, U \cup V) & \xrightarrow{\partial} & H_i(U \cup V, V) & \xrightarrow{\text{incl}_*} & H_i(M, V) \\ & & \uparrow \cong & & \uparrow \text{incl}_* \\ & & H_i(U) & \xrightarrow{\text{incl}_*} & H_i(M) \end{array}$$

$i \geq n+1$   $\Rightarrow$  top left & right term zero by Lemma 3  $\Rightarrow$  top middle zero  $\Rightarrow$

$$H_i(U) = 0 \Rightarrow [z] = 0 \in H_i(U) \Rightarrow [z] = 0 \in H_i(M). \quad \checkmark$$

$i = n$   $[z]$  defines a section  $M \rightarrow M_{\mathbb{R}}$  by

$$x \mapsto (x, \text{image of } [z] \text{ under } H_n(M) \rightarrow H_n(M \setminus x))$$

Pick  $x_0 \in V$ . Then  $x_0 \mapsto (x_0, 0)$ .  $M$  connected  $\Rightarrow$

$\exists$  unique section  $M \rightarrow M_{\mathbb{R}}$  with  $x_0 \mapsto (x_0, 0) \Rightarrow$  the section defined by  $[z]$  sends  $x \mapsto (x, 0)$  for all  $x$ .

Lemma 3  $\Rightarrow [z] = 0 \in H_i(M, V)$ . Top left term zero  $\Rightarrow$

$$[z] = 0 \in H_i(U \cup V, V) \Rightarrow [z] = 0 \in H_i(U) \Rightarrow [z] = 0 \in H_i(M) \quad \square$$

## ⑧ Poincaré Duality

### Sneak preview

#### Theorem 4 (Poincaré duality)

Let  $M$  be a closed  $\mathbb{R}$ -oriented  $n$ -dim manifold. Then for all  $k \in \mathbb{Z}$ ,

$$H^k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R}).$$

#### Theorem 7 $M^n$ compact (potentially with $\partial$ ) $\Rightarrow$

$H_*(M; \mathbb{R})$  is a finitely generated  $\mathbb{R}$ -module.

**Proof idea** Use that  $M \simeq$  some finite  $\Delta$ -complex

(Hatcher A.8, A.9 p. 527)

□

#### Corollary 8 $M^n$ closed, $\mathbb{K}$ -orientable for a field $\mathbb{K}$

$$H_k(M; \mathbb{K}) \underset{\text{UCT}}{\cong} H^k(M; \mathbb{K}) \underset{\text{PD}}{\cong} H_{n-k}(M; \mathbb{K})$$

#### Corollary 9 $M^n$ closed, $n$ odd $\Rightarrow \chi(M) = 0$ .

**Def** Let  $X$  be a top. space,  $R$  a commutative unital ring,  
 $\sigma \in C_n(X; R)$ ,  $\varphi \in C^k(X; R)$  with  $k \leq n$ .

Then the **cap product** is

$$\sigma \frown \varphi = \varphi(\sigma|_{[e_0, \dots, e_k]}) \sigma|_{[e_k, \dots, e_n]} \in C_{n-k}(X; R)$$

### Proposition 1

(1) Linear extension gives an  $R$ -bilinear map

$$C_n(X; R) \times C^k(X; R) \longrightarrow C_{n-k}(X; R)$$

(2)  $\sigma \frown \varepsilon = \sigma$  for  $\varepsilon \in C^0(X; R)$ ,  $\varepsilon(\tau) = 1 \forall \tau$ .

(3)  $(\sigma \frown \varphi) \frown \psi = \sigma \frown (\varphi \cup \psi)$ .

**Pf** Exercise

□

**Proposition 2**  $(-1)^k d(\sigma \frown \varphi) = (d\sigma) \frown \varphi - \sigma \frown d\varphi$

**Pf**  $d(\sigma \frown \varphi) = \sum_{i=k}^n \varphi(\sigma|_{[e_0, \dots, e_k]}) (-1)^{i+k} \sigma|_{[e_k, \dots, \hat{e}_i, \dots, e_n]}$

$$\begin{aligned} (d\sigma) \frown \varphi &= \sum_{j=0}^k (-1)^j \varphi(\sigma|_{[e_0, \dots, \hat{e}_j, \dots, e_{k+1}]}) \sigma|_{[e_{k+1}, \dots, e_n]} \\ &+ \sum_{l=k+1}^n (-1)^l \varphi(\sigma|_{[e_0, \dots, e_k]}) \sigma|_{[e_k, \dots, \hat{e}_l, \dots, e_n]} \end{aligned}$$

$$\sigma \frown (d\varphi) = \sum_{m=0}^{k+1} (-1)^m \varphi(\sigma|_{[e_0, \dots, \hat{e}_m, \dots, e_{k+1}]}) \sigma|_{[e_{k+1}, \dots, e_n]}$$

□

**Proposition 3** (1) cycle  $\cap$  cocycle = cycle

(2) boundary  $\cap$  cocycle = boundary

(3) cycle  $\cap$  coboundary = boundary

(4) For  $[c] \in H_m(X; R)$ ,  $[\varphi] \in H^k(X; R)$ ,

$$[c] \cap [\varphi] := [c \cap \varphi] \in H_{m-k}(X; R)$$

is a well-defined  $R$ -bilinear map.

(5)  $X$  path-connected,  $\delta: H_0(X; R) \rightarrow R$  the iso  $[\sigma] \mapsto 1$ ,

$[c] \in H_m(X; R)$ ,  $[\varphi] \in H^m(X; R)$ , then

$$\delta([c] \cap [\varphi]) = \varphi(c) = \text{ev}([\varphi])([c])$$

**Proof:** Exercise.

**Theorem 4 (Poincaré duality)**

Let  $M$  be a closed  $R$ -oriented  $n$ -dim manifold. Then for all  $k \in \mathbb{Z}$ ,

$$PD: H^k(M; R) \longrightarrow H_{n-k}(M; R)$$

$$PD([\varphi]) = [M] \cap [\varphi]$$

is an isomorphism.

Before we dive into the consequences of PD, here are two more properties of the cap product.

**Prop 5 (Naturality of cap)**  $f: X \rightarrow Y$  cont.,  $a \in C_m(X)$ ,  $\varphi \in C^k(Y)$

$$f_c(a \cap f^c \varphi) = (f_c a) \cap \varphi$$

**Proof** Exercise.

**Remark 6** Similarly as for the cup, one may define a **relative cap**

$$\frown : H_n(X, A \cup B; R) \times H^k(X, A; R) \longrightarrow H_{n-k}(X, B; R)$$

using that  $C_*(A+B) \hookrightarrow C_*(A \cup B)$  induces isos

$$H_*(X, A+B) \longrightarrow H_*(X, A \cup B).$$

We'll prove PD, but first, let us harvest some implications.

Let us take the following for granted.

**Theorem 7**  $M^n$  compact (potentially with  $\partial$ )  $\Rightarrow$   
 $H_*(M; R)$  is a finitely generated  $R$ -module.

**Proof idea** Use that  $M \simeq$  some finite  $\Delta$ -complex  
 (Hatcher A.8, A.9 p. 527)  $\square$

**Corollary 8**  $M^n$  closed,  $\mathbb{K}$ -orientable for a field  $\mathbb{K}$

$$H_k(M; \mathbb{K}) \cong H^k(M; \mathbb{K}) \cong H_{n-k}(M; \mathbb{K}) \cong H^{n-k}(M; \mathbb{K})$$

**Proof** Since  $H_*(M)$  f.g. by Thm 7:

$$\begin{aligned} \dim H_k(M; \mathbb{K}) &\stackrel{\text{UCT hom}}{=} \# \mathbb{Z}\text{-summands of } H_k(M) + \\ &\quad \# \mathbb{Z}_p\text{-summands of } H_k(M) \text{ and } H_{k-1}(M) \\ &\stackrel{\text{UCT cohom}}{=} \dim H^k(M; \mathbb{K}) \end{aligned}$$

$p = \text{char } \mathbb{K}$

This proves the first iso. The second is PD.  $\square$

**Corollary 9**  $M^n$  closed,  $n$  odd  $\Rightarrow \chi(M) = 0$ .

**Proof**  $\chi(M) = \sum_{k=0}^n (-1)^k \dim H_k(M; \mathbb{F}_2) \quad n = 2m+1$

$$= \sum_{k=0}^m (-1)^k \dim H_k(M; \mathbb{F}_2) + (-1)^{2m+1-k} \dim H_{2m+1-k}(M; \mathbb{F}_2) = 0 \quad \square$$

**Proposition 10**  $M^n$  connected, closed, oriented st  $H_*(M)$  is free.

Then  $\cup : H^k(M) \times H^{n-k}(M) \longrightarrow H^n(M) \cong H_0(M) \cong \mathbb{Z}$   
PD  $\delta : [\sigma] \mapsto 1$   
 is non-singular, ie

$$H^k(M) \longrightarrow \text{Hom}(H^{n-k}(M), \mathbb{Z})$$

$$[\varphi] \mapsto ([\psi] \mapsto \delta(\text{PD}([\psi] \cup [\varphi])))$$

is an iso.

**Proof**  $H_*(M)$  free by assumption  $\Rightarrow \text{Ext}(H_{k-1}(M), \mathbb{Z})$  is trivial

$\Rightarrow$  ev is iso. So we have isos

$$H^k(M) \xrightarrow{\text{ev}} \text{Hom}(H_k(M), \mathbb{Z})$$

$$\xrightarrow{\text{PD}^*} \text{Hom}(H^{n-k}(M), \mathbb{Z})$$

Just need to check that their composition equals the desired homomorphism  $[\varphi] \mapsto ([\psi] \mapsto \delta(\text{PD}([\psi] \cup [\varphi])))$ .

Let  $[\varphi] \in H^k(M)$ ,  $[\psi] \in H^{n-k}(M)$ . Then

$$\begin{aligned} \text{PD}^*(\text{ev}([\varphi]))([\psi]) &= \text{ev}([\varphi])(\text{PD}([\psi])) \\ &= \text{ev}([\varphi])([M] \cap [\psi]) \\ &= \delta([M] \cap [\psi] \cap [\varphi]) \\ &= \delta([M] \cap ([\psi] \cup [\varphi])) \\ &= \delta(\text{PD}([\psi] \cup [\varphi])) \end{aligned}$$

□

**Remark 11** (1)  $M^n$  closed, orientable,  $H_*(M)$  free

$$\Rightarrow H^k(M) \cong H_k(M) \cong H^{n-k}(M) \cong H_{n-k}(M).$$

(2) A bilinear form  $b: \mathbb{Z}^m \times \mathbb{Z}^m \rightarrow \mathbb{Z}$  is non-singular

$$\Leftrightarrow \mathbb{Z}^m \longrightarrow \text{Hom}(\mathbb{Z}^m, \mathbb{Z}), \quad x \mapsto (y \mapsto b(x, y))$$

is an iso

$\Leftrightarrow \forall$  primitive  $x \in \mathbb{Z}^m$  (i.e.  $x$  not divisible by integers  $\geq 2$ ,  
or equivalently:  $x$  can be extended to a basis)

$$\exists y \in \mathbb{Z}^m \text{ s.t. } b(x, y) = 1.$$

**Theorem 12**  $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$  with  $\deg x = 2$ .

**Proof** By induction over  $n$ . For  $n=0$ ,  $\mathbb{C}P^0 \cong \{*\}$ ,  $H^*(\{*\}) \cong \mathbb{Z}$

For  $n=1$ ,  $\mathbb{C}P^1 \cong S^2$  and  $H^*(S^2) \cong \mathbb{Z}[x]/(x^2)$ . Assume  $n \geq 2$

and  $H^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n)$ . The embedding  $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$  induces isos on  $H^k$  for  $k < 2n$  (evident from CW-structure).

Let  $x$  be a generator of  $H^2(\mathbb{C}P^n)$ . By naturality of  $\smile$  and the induction hypothesis,  $x^k$  generates  $H^{2k}(\mathbb{C}P^n)$  for  $k < 2n$ .

It just remains to show that  $x^n$  generates  $H^{2n}(\mathbb{C}P^n)$ .

Since  $\smile$  is non-singular (Prop 10) and  $x^k$  is primitive (since it is a generator), by Prop 11(2)  $\Rightarrow \exists y \in H^{2n-2}(\mathbb{C}P^n)$  st

$x \smile y$  generates  $H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$ . Since  $H^{2n-2}(\mathbb{C}P^n) = \mathbb{Z}x^{n-1} \Rightarrow$

$\exists m \in \mathbb{Z}$  with  $y = mx^{n-1}$ . Since  $x \smile y = mx^n$  generates  $H^{2n}(\mathbb{C}P^n)$

$\Rightarrow m = \pm 1 \Rightarrow y = \pm x^{n-1} \Rightarrow x^n$  generates  $H^{2n}(\mathbb{C}P^n)$ .  $\square$

**Remark 13** Note that  $[\varphi] \in TH^k(X)$

$\Rightarrow$  for all  $[\varphi] \in H^k(X)$  we have  $[\varphi] \smile [\varphi] \in TH^{k+l}(X)$ .

So  $\smile$  induces  $\underline{FH}^k(X) \times \underline{FH}^l(X) \longrightarrow \underline{FH}^{k+l}(X)$ .

recall:  $FA = A/TA$  is the "free part" of an ab. group  $A$ .

For  $M$  closed, connected, oriented,

$\smile : \underline{FH}^k(X) \times \underline{FH}^{n-k}(X) \longrightarrow \underline{FH}^n(X) \cong \mathbb{Z}$ .

is non-singular (similar proof as for Prop 10).



# Proposition 14 (Ev for other rings)

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Let  $C_\bullet$  be a chain complex,  $R$  a commutative unital ring, and  $M$  an  $R$ -module.

(1) There is an isomorphism of cochain complexes over  $R$

$$i: \text{Hom}_{\mathbb{Z}}(C_\bullet, M) \longrightarrow \text{Hom}_R(C_\bullet \otimes_{\mathbb{Z}} R, M)$$

$$\varphi \longmapsto (c \otimes r \longmapsto \varphi(c)r)$$

with inverse  $i^{-1}$ :

$$(c \mapsto \varphi(c \otimes 1)) \longleftarrow \varphi.$$

$$(2) \quad \text{ev}_R: H^n(C; M) \longrightarrow \text{Hom}_R(H_n(C; R), M)$$

$$[\varphi] \longmapsto ([\alpha] \longmapsto i(\varphi)(\alpha))$$

is a well-defined  $R$ -linear map.

$$(3) \quad H^n(C; M) \xrightarrow{\text{ev}} \text{Hom}_{\mathbb{Z}}(H_n(C), M)$$

$$\text{ev}_R \searrow \text{Hom}_R(H_n(C; R), M) \xrightarrow{f \mapsto ([\alpha] \mapsto f([\alpha \otimes 1_R])}$$

commutes.

(4) If  $R$  is a field, then  $\text{ev}_R$  is an isomorphism.

**Proof** (1) To check:  $\ast i_n(\varphi)$  is an  $R$ -homom.  $C_n \otimes R \rightarrow M$

$\ast i_n$  is an  $R$ -homom. at each homological degree

$\ast i$  is a cochain map

$\ast i_n^{-1}$  is an  $\mathbb{Z}$ -homom.  $C_n \rightarrow M$

$\ast i \circ i^{-1}, i^{-1} \circ i$  are identity maps.

(2) To check:

$$i(\varphi)(\alpha) = 0 \text{ if } \begin{cases} \alpha \text{ boundary, } \varphi \text{ is cocycle} \\ \alpha \text{ is cycle, } \varphi \text{ is coboundary} \end{cases} \text{ or}$$

(3) By def of  $\text{ev}$  and  $\text{ev}_R$ .

(4) Same proof as UCT, using  $\text{Ext}_R^1$  is always 0 since all  $R$ -modules are free.  $\square$

**Prop 15**  $M^n$  closed, connected,  $\mathbb{K}$ -oriented for a field  $\mathbb{K}$

Then  $H^*(M; \mathbb{K})$  is a Poincaré algebra of formal dim.  $n$ .

**Proof** (i)  $H^j(M; \mathbb{K}) = 0$  for  $j > n$

since  $H^j(M; \mathbb{K}) \cong H_{n-j}(M; \mathbb{K}) \cong 0$  since  $n-j < 0$ . ✓

(ii)  $H^n(M; \mathbb{K}) \cong \mathbb{K}$  since  $H^n(M; \mathbb{K}) \xrightarrow[\cong]{PD} H_0(M; \mathbb{K}) \xrightarrow[\cong]{S} \mathbb{K}$ . ✓

(iii) The  $\mathbb{K}$ -bilinear pairing

$$\cup: H^j(M; \mathbb{K}) \times H^{n-j}(M; \mathbb{K}) \rightarrow H^n(M; \mathbb{K}) \cong \mathbb{K}$$

is non-singular  $\Leftrightarrow$  the adjoint homom.

$$H^j(M; \mathbb{K}) \rightarrow \text{Hom}_{\mathbb{K}}(H^{n-j}(M; \mathbb{K}), \mathbb{K})$$

$$[\varphi] \mapsto ([\varphi] \mapsto S(PD([\varphi] \cup [\varphi])))$$

is an iso. Show (similarly as in Prop 10) that the adjoint equals the composition of

$$\begin{aligned} H^j(M; \mathbb{K}) &\xrightarrow{e\omega_{\mathbb{K}}} \text{Hom}_{\mathbb{K}}(H_j(M; \mathbb{K}), \mathbb{K}) \\ &\xrightarrow{PD^*} \text{Hom}_{\mathbb{K}}(H^{n-j}(M; \mathbb{K}), \mathbb{K}) \quad \square \end{aligned}$$

**Corollary 16**  $H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$  with  $\deg x = 1$ .

**Proof** Same as Thm 12, using Prop 15. □

**Long Example 17**  $M^4$  closed, simply connected.

What do we know about  $H_*(M), H^*(M)$ ?

Simply connected  $\Rightarrow$  connected  $\Rightarrow H_0 \cong H^0 \cong \mathbb{Z}$

— " —  $\Rightarrow$  orientable  $\Rightarrow H_4 \cong H^4 \cong \mathbb{Z}$  and PD holds

— " —  $\Rightarrow H_1 = 0$  by Hurewicz Thm  $\Rightarrow H^3 = 0$  by PD

UCT  $\Rightarrow H^1 \cong FH_1 \oplus TH_0 \cong 0$ . PD  $\Rightarrow H_3 \cong 0$ .

UCT  $\Rightarrow H^2 \cong FH_2 \oplus TH_1 \cong FH_2$ , so  $H^2$  is torsion free and thus free (because  $H_*$  f.g. by Thm 7). PD  $\Rightarrow H_2 \cong H^2$ .

So  $H_*(M), H^*(M)$  are determined except for  $\text{rk } H_2(M) \in \{0, 1, 2, \dots\}$

What about the cohomology ring?  $\cup: H^2(M) \times H^2(M) \rightarrow H^4(M)$

is non-singular (Prop 10) and symmetric (since

$[c_1] \cup [c_2] = (-1)^{2 \cdot 2} [c_2] \cup [c_1]$ ). Pick an orientation of  $M$ :

that yields an isomorphism  $H^4(M) \rightarrow \mathbb{Z}$  (via  $H^4 \xrightarrow{\text{PD}} H_0 \xrightarrow{\cong} \mathbb{Z}$ )

Pick a basis for  $H^2(M)$ , i.e. an iso  $H^2(M) \cong \mathbb{Z}^m$ . Then  $\cup$  becomes

a non-singular symmetric bilinear form  $\mathbb{Z}^m \times \mathbb{Z}^m \rightarrow \mathbb{Z}$ .

Such a form may be written as a matrix  $A \in \mathbb{Z}^{m \times m}$  with

$$v \cup w = v^t A w \text{ for } v, w \in \mathbb{Z}^m.$$

Eg for  $M = \mathbb{C}P^2$ , we find  $A = (1)$  or  $A = (-1)$ , depending on the orientation on  $\mathbb{C}P^2$ .

— Non-singular  $\Rightarrow \det A = \pm 1$ .

— Symmetric  $\Rightarrow A^t = A$ . Picking a different basis for  $H^2(M)$  transforms  $A$  into  $T^t A T$  for  $T \in \mathbb{Z}^{m \times m}$  with  $\det T = \pm 1$ .

Picking the opposite orientation for  $M$  transforms  $A$  into  $-A$ .

Long Example 17 (cont'd)  $M^4$  closed, simply connected.

Shown last time:  $H_0 \cong H_4 \cong \mathbb{Z}$ ,  $H_1 \cong H_3 = 0$ ,  $H_2 = \mathbb{Z}^m$  for some  $m \geq 0$ .

What about the cohomology ring?  $\smile: H^2(M) \times H^2(M) \longrightarrow H^4(M)$

is non-singular (Prop 10) and symmetric (since

$[c_1] \smile [c_2] = (-1)^{2 \cdot 2} [c_2] \smile [c_1]$ ). Pick an orientation of  $M$ :

that yields an isomorphism  $H^4(M) \rightarrow \mathbb{Z}$  (via  $H^4 \xrightarrow{PD} H_0 \xrightarrow{\int} \mathbb{Z}$ )

Pick a basis for  $H^2(M)$ , i.e. an iso  $H^2(M) \cong \mathbb{Z}^m$ . Then  $\smile$  becomes a non-singular symmetric bilinear form  $\mathbb{Z}^m \times \mathbb{Z}^m \rightarrow \mathbb{Z}$ .

Such a form may be written as a matrix  $A \in \mathbb{Z}^{m \times m}$  with

$$v \smile w = v^t A w \text{ for } v, w \in \mathbb{Z}^m.$$

Eg for  $M = \mathbb{C}P^2$ , we find  $A = (1)$  or  $A = (-1)$ , depending on the orientation on  $\mathbb{C}P^2$ .

$\smile$  Non-singular  $\Rightarrow \det A = \pm 1$ .

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Picking the opposite orientation for  $M$  transforms  $A$  into  $-A$ .

If  $M \cong N$  via a map  $f: M \rightarrow N$

$$\text{call } f \begin{cases} \text{orientation-preserving (op)} & \text{if } \deg f = 1 \\ \text{-reversing} & \text{if } \deg f = -1 \end{cases}$$

then  $A_M = (\deg f) \cdot T^t A_N T$  for some  $T$ .

Ex  $\mathbb{C}P^2$  and  $\overline{\mathbb{C}P^2}$  are not o.p. hom. equiv.

since  $(1) \neq T^t (-1) T$  for  $T = (\pm 1)$ .

Thm (Whitehead) The converse holds:

$$M \underset{\text{o.p.}}{\cong} N \text{ iff } A_M = T^t A_N T.$$

### ③ Cohomology with compact support & Proof of PD

Proof idea for PD: induction over number of charts, using Mayer-Vietoris to glue charts together. Problem: Union of charts may be non-compact.

Solution: Define a new cohomology theory  $H_c^k$  st  $H_c^k \cong H^k$  if  $M$  compact, and extend PD:

**Theorem 1** (PD without compactness assumption)  $R$  commutative ring with 1,  $M^n$  be oriented. Then we have an isom (to be defined later)

$$PD: H_c^k(M; R) \longrightarrow H_{n-k}(M; R).$$

**Motivation for  $H_c^k$**   $X$  a **locally finite**  $\Delta$ -complex, ie every  $k$ -simplex is face of only finitely many  $(k+1)$ -simplexes.

Let the **simplicial cochain complex with compact support** be

$$C_{c\Delta}^k(X) := \{ \varphi \in C_{\Delta}^k(X) \mid \varphi(\sigma) = 0 \text{ except for finitely many } k\text{-simplexes } \sigma \in X \}$$

Note  $C_{c\Delta}^{\bullet} \subseteq C_{\Delta}^{\bullet}$  is a subcomplex.

Eg  $X = \dots \xrightarrow{e_{-1}} \underset{v_{-1}}{\circ} \xrightarrow{e_0} \underset{v_0}{\circ} \xrightarrow{e_1} \underset{v_1}{\circ} \xrightarrow{e_2} \underset{v_2}{\circ} \dots$   $de_i = e_{i+1} - e_i$

Since  $X \cong \mathbb{R} \Rightarrow H_0^{\Delta}(X) \cong H_{\Delta}^0(X) \cong \mathbb{Z}$ ,  $H_k^{\Delta}(X) \cong H_{\Delta}^k(X) \cong 0 \quad \forall k \geq 1$ ,  
so PD doesn't hold for  $\mathbb{R}$ . Let us check that it does when using  $H_{c\Delta}^k$ !

$$v^i(v_j) = e^i(e_j) = \delta_{ij}.$$

$$C_{c\Delta}^0(X) = \bigoplus_{i \in \mathbb{Z}} v^i \xrightarrow{d^0} C_{c\Delta}^1(X) = \bigoplus_{i \in \mathbb{Z}} e^i$$

$$v^i \longmapsto e^{i-1} - e^i$$

since  $d^0(v^i)(e_j) = v^i(d_1(e_j)) = v^i(v_{j+1} - v_j) = \delta_{i,j+1} - \delta_{i,j}$

So  $\ker d^0 = 0$  and  $\operatorname{coker} d^0 \cong \mathbb{Z}$  generated by  $[e^i]$  for any  $i$ .  
 $\Rightarrow H_{\Delta}^0(X) \cong 0$ ,  $H_{\Delta}^1(X) \cong \mathbb{Z}$  and PD holds.

**Def**  $X$  top. space,  $A$  ab. group. Let the singular cochain complex with compact support of  $X$  with coefficients in  $A$  be

$$C_c^k(X; A) := \left\{ \varphi \in C^k(X; A) \mid \exists \text{ compact } K \subseteq X \text{ s.t. } \varphi(\sigma) = 0 \text{ for all } \sigma: \Delta^k \rightarrow X \text{ with } \text{im}(\sigma) \cap K = \emptyset \right\}$$

Note  $C_c^k \subseteq C^k$  is a subcomplex, because  $\varphi \in C_c^k(X; A) \Rightarrow d^k \varphi(\sigma) = \varphi(d_{k+1} \sigma) = 0$  for  $\sigma: \Delta^{k+1} \rightarrow X$  with  $\text{im}(\sigma) \cap K = \emptyset$ , since  $\text{im}(d\sigma) \subseteq \text{im}(\sigma) \Rightarrow \text{im}(d\sigma) \cap K = \emptyset$ .

$H_c^k(X; A) :=$  cohomology of  $C_c^\bullet(X; A)$  is called singular cohomology with compact support of  $X$  with coefficients in  $A$ .

**Remark 2**  $C_c^k(X; A) = C^k(X; A)$  if  $X$  is compact (take  $K = X$ ).

**Def** Let  $I$  be a set partially ordered by  $\leq$  (i.e.  $\leq$  is reflexive, antisymmetric and transitive). If  $\forall \alpha, \beta \in I \exists \gamma \in I$  with  $\alpha \leq \gamma, \beta \leq \gamma$  then  $(I, \leq)$  is called a directed set.

eg subsets of a fixed set  $X$  ordered by inclusion, or open subsets of  $X$ , or compact subsets of  $X$ .