Algebraic Topology II (FS '24, ETHZ) 14 Feb

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Alg Top I Top. Space X

$$\rightsquigarrow$$

Singular Chain Complex $C(X) = \cdots \to C_1(X) \xrightarrow{d_1} C_0(X) \to 0$

$$\rightsquigarrow$$

Homology groups $H_i(X)$

Alg Top II Spice up $C(X)$ before taking homology to get more sensitive invariants and more geom. applications

Topics:

* Homology with Coefficients (for abelian groups M define chain complex $C(X) \otimes M$ with homology groups $H_i(X; M)$)
* Cohomology (cochain complex $\operatorname{Hom}(C(X), M)$ with cohomology groups $H^i(X; M)$)
* Poincaré Duality for compact n-dim manifolds X ($H_i(X; M) \cong H^{n-i}(X; M)$, leading to intersection forms $H_{n/2}(X) \times H_{n/2}(X) \to \mathbb{Z}$ for even n)

Color Scheme:

Sections, Date

Def / Thm / Proof etc.

Newly defined terms

References

Corrections

(1) Tensor Products of modules

(Spanier: Intro, Sec 4 & Ch 5 Sec 1; Hatcher Sec 3.2 / A Künneth formula; Atiyah–MacDonald Ch 2 / Tensor Product of modules)

Let R be a commutative ring with 1 (after this section only $R = \mathbb{Z}$).

Prop 1 Let M, N be R-modules. Then there exists an R-module T and a bilinear map $\mu : M \times N \longrightarrow T$ such that:

For all R-modules K and bilinear maps $f : M \times N \longrightarrow K$ there is a unique homomorphism $g : T \to K$ with $g \circ \mu = f$.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & K \\ T \arrow[ur, dashed, "\exists ! g"'] & \end{tikzcd}$$

Proof $U :=$ free R-module with basis the set $M \times N$.

$I :=$ submodule of U generated by

$$\{ (\lambda x + x', y) - \lambda(x,y) - (x',y) \mid \lambda \in R,\ x, x' \in M,\ y \in N \}$$
$$\cup \{ (x, \lambda y + y') - \lambda(x,y) - (x,y') \mid \lambda \in R,\ x \in M,\ y, y' \in N \}$$

Let $T = U/I$ and $\mu : M \times N \longrightarrow T$, $\mu(x,y) = [(x,y)]$

Check that μ is bilinear! Now let $f : M \times N \to K$ as above be given.

Check <u>existence of g</u>:

Let $\tilde{g} : U \to K$ be the homomorphism with $\tilde{g}((x,y)) = f(x,y)$.

Check that $I \subseteq \ker \tilde{g} \Rightarrow \tilde{g}$ induces $g : T \longrightarrow K$.

We have $g(\mu(x,y)) = g([(x,y)]) = \tilde{g}((x,y)) = f(x,y)$ ✓

Check <u>uniqueness of g</u>:

If $g' : T \to K$ with $g' \circ \mu = f$, then $g'([(x,y)]) = g([(x,y)])$ for all $x \in M$, $y \in N$. But such $[(x,y)]$ generate $T \Rightarrow g = g'$ ✓

$\square$

Prop 2 If $\mu: M \times N \to T$ and $\mu': M \times N \to T'$ both satisfy the condition in Prop 1, then there is a unique isomorphism $\varphi: T \to T'$ such that $\varphi \circ \mu = \mu'$.

$$\begin{tikzcd} & M \times N \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] \arrow[d, phantom, "\circlearrowleft"] & \\ T \arrow[rr, "\varphi"'] & & T' \end{tikzcd}$$

Proof By assumption (existence of g), $\exists\, \varphi: T \to T'$ with $\varphi \circ \mu = \mu'$ and $\exists\, \psi: T' \to T$ with $\psi \circ \mu' = \mu$. Then $\psi \circ \varphi: T \to T$ with $\psi \circ \varphi \circ \mu = \mu$. By assumption (uniqueness of g) $\Rightarrow \psi \circ \varphi = id_T$. Similarly $\varphi \circ \psi = id_{T'}$. $\square$

Def T as in Prop 1 is called the tensor product of M and N over R, written $M \otimes_R N$. Drop R if there is no ambiguity. Write $x \otimes y = \mu(x,y) \in M \underset{R}{\otimes} N$.

Notation $\times$ and $\oplus$ is the same for finitely many modules.

Prop 3 (1) $\exists$ iso $M \otimes N \to N \otimes M$ with $x \otimes y \mapsto y \otimes x$.

(2) $\exists$ iso $(M \oplus N) \otimes K \to (M \otimes K) \oplus (N \otimes K)$ with $(x,y) \otimes z \mapsto (x \otimes z) + (y \otimes z)$

(3) $I \subseteq R$ ideal $\Rightarrow \exists$ iso $(R/I) \otimes M \to M/IM$ with $r \otimes m \mapsto [rm]$

Rmk 4 Special case of (3): Iso $R \otimes M \to M$, $r \otimes m \mapsto rm$.

Proof of Prop 3 (1)

$$\begin{tikzcd} M \oplus N \arrow[rr, "h \cdot (x,y) \mapsto (y,x)"] \arrow[d, "\mu_{M,N}"'] & \circlearrowleft & N \oplus M \arrow[d, "\mu_{N,M}"] \\ M \otimes N \arrow[rr, dashed, "\exists ! \varphi"'] & & N \otimes M \end{tikzcd}$$

Let $h \cdot M \oplus N \to N \oplus M$ be the homom. with $(x,y) \longmapsto (y,x)$.

Then $\mu_{N,M} \circ h : M \oplus N \longrightarrow N \otimes M$ is bilinear.

By the universal property of $\otimes$, $\exists$ $\varphi : M \otimes N \longrightarrow N \otimes M$ with

$\varphi \circ \mu_{M,N} = \mu_{N,M} \circ h$, ie $\varphi(x \otimes y) = y \otimes x$.

Let ψ be the analoguous homo with M, N switched $\Rightarrow$

φ, ψ are mutually inverse homomorphisms.

In the lecture, a similar (but incorrect) proof was given, based on the erroneous assumption that h is bilinear (it is, in fact, linear).

Proof of (2) – (4): Exercises. $\square$

Rmk 5 Using Prop 3, we can calculate $M \otimes_{\mathbb{Z}} N$ for all finitely generated abelian groups M, N.

Example 6 $\mathbb{Z}^2 \otimes \mathbb{Z}^2 = (\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}^4$

So $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ is free with basis $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$.

Careful! Not every element of $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ is of the form $x \otimes y$,

eg $e_1 \otimes e_1 + e_2 \otimes e_2$ isn't (and isn't equal to $(e_1 + e_2) \otimes (e_1 + e_2)$).

Rmk 7 (1) Every element of $M \otimes N$ is equal to $\sum_{i=1}^{n} x_i \otimes y_i$ for some finite n, $x_i \in M$, $y_i \in N$.

(2) $(\lambda x) \otimes y = x \otimes (\lambda y)$

(3) $(x + x') \otimes y = x \otimes y + x' \otimes y$

Prop 8 $f: M \to N$, $f': M' \to N'$ R-module homoms.

(1) $\exists$ homo $f \otimes f': M \otimes M' \longrightarrow N \otimes N'$ with $x \otimes x' \mapsto f(x) \otimes f'(x')$.

(2) $(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$.

(3) $(f+g) \otimes f' = f \otimes f' + g \otimes f'$ and similarly in second factor.

Pf (1) Induced by the bilinear map $M \times M' \longrightarrow N \otimes N'$,

$$(x, x') \longmapsto f(x) \otimes f'(x').$$

(2), (3) Check that $x \otimes x'$ has the same image under both maps. $\square$

Prop 9 M an abelian group, S a commutative ring. Then $M \underset{\mathbb{Z}}{\otimes} S$ carries an S-module structure given by $s \cdot (x \otimes t) = x \otimes st$.

For homom $f: M \to N$ and S-homom $g: S \to S$,

$f \otimes g: M \otimes S \longrightarrow N \otimes S$ is an S-homom.

Proof: Exercise (careful: why is the function $x \otimes t \mapsto x \otimes st$ well-def?).

Category theory intermezzo Weibel Sec 1.1, 1.2

Reminder A Category $\mathcal{C}$ consists of a class $|\mathcal{C}|$ of objects, for all $X, Y \in |\mathcal{C}|$ a set $\mathcal{C}(X,Y)$ of morphisms with a distinguished identity morphism $1_X \in \mathcal{C}(X,X)$, and composition functions $\circ: \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ such that $(f \circ g) \circ h = f \circ (g \circ h)$ and $f \circ 1_X = 1_X \circ f = f$.

A (covariant) functor $F: \mathcal{C} \to \mathcal{D}$ consists of functions $|\mathcal{C}| \to |\mathcal{D}|$ and $\mathcal{C}(X,Y) \longrightarrow \mathcal{D}(FX, FY)$ with $F(f \circ g) = Ff \circ Fg$ and $F1_X = 1_{FX}$. For a contravariant functor, one has instead $\mathcal{C}(X,Y) \to \mathcal{D}(FY, FX)$ and $F(f \circ g) = Fg \circ Ff$.

Def A **preadditive category** $\mathcal{C}$ is a category with abelian group structures on $\mathcal{C}(X,Y)$, such that compositions are bilinear. A functor F between preadditive $\mathcal{C} \to \mathcal{D}$ is **additive** if the functions $\mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ (or $\to \mathcal{D}(FY, FX)$ if F is contravariant) are linear.

Examples 10 R commutative ring with 1.

(1) The category R-Mod of R-modules and R-homomorphisms is preadditive.

(2) Chain complex over a preadditive category $\mathcal{E}$: sequence of $C_0, C_1, \ldots \in |\mathcal{E}|$ and morphisms $d_1 : C_1 \to C_0$, $d_2 : C_2 \to C_1, \ldots$ with $d_i \circ d_{i+1} = 0$.
The cat. $Ch(\mathcal{E})$ of $\mathcal{E}$-chain complexes and chain maps is again preadditive. Chain maps $f : C \to C'$ are sequences $f_0, f_1, \ldots$ with $f_i \in \mathcal{E}(C_i, C_i')$ and $f_i \circ d_{i+1} = d'_{i+1} \circ f_{i+1}$ for all $i \geq 0$.

(3) Cat of Top spaces $\longrightarrow Ch(\mathbb{Z}\text{-Mod})$,
$X \longmapsto C(X)$, $f \longmapsto f_c$ is a functor (AlgTop I)

(3') Refinement of (3): Functor

$$\underbrace{\text{Cat of Pairs of Top spaces}} \longrightarrow Ch(\mathbb{Z}\text{-Mod})$$
$$(X,A) \longmapsto C(X,A),$$
$$f \longmapsto f_c$$

Objects: (X,A) with X Top space, $A \subseteq X$.
Morphisms $f : (X,A) \to (Y,B)$: $f : X \to Y$ cont. with $f(A) \subseteq B$.

(4) $Ch(R\text{-Mod}) \longrightarrow R\text{-Mod}$, $C \longmapsto H_i(C) := \ker d_i / \operatorname{im} d_{i+1}$

$f \longmapsto f_*$ are additive functors for each fixed $i \geq 0$.

(5) Composing (3′) and (4) gives functors

Pairs of top spaces $\longrightarrow \mathbb{Z}\text{-Mod}$, $(X,A) \longmapsto H_i(X,A)$,

$f \longmapsto f_*$.

(6) M a fixed R-module. Then $R\text{-Mod} \longrightarrow R\text{-Mod}$,

$N \longmapsto N \otimes_R M$, $\quad f \longmapsto f \otimes \mathrm{id}_M$ also written as $f \otimes M$

is an additive functor! (see Prop 8)

(6′) $\mathbb{Z}\text{-Mod} \longrightarrow S\text{-Mod}$,

$M \longmapsto M \otimes_{\mathbb{Z}} S$, $\quad f \longmapsto f \otimes \mathrm{id}_S$

is another additive functor (see Prop 9)

② Homology with coefficients

Spanier 5.1, Hatcher 2.2

X top. space, $A \subseteq X$, M an abelian group.

Prop 1
$$\cdots \xrightarrow{d_2 \otimes \mathrm{id}_M} C_1(X,A) \otimes M \xrightarrow{d_1 \otimes \mathrm{id}_M} C_0(X,A) \otimes M \longrightarrow 0$$

is a chain complex.

Proof postponed.

Def We call the complex in Prop 1 the chain complex of (X,A) with coefficients in M, denoted by $C(X,A) \otimes M$. We call $H_i(C(X,A) \otimes M)$ the i-th homology group with coefficients in M, denoted by $H(X,A;M)$.

Rmk 2 $C(X,A) \otimes \mathbb{Z}$ is naturally isomorphic to $C(X,A)$.

Goal Chain complexes & homology groups with any coefficients M have all the good properties proven for $\mathbb{Z}$ coefficients in Alg Top I.

Rmk 4 Recall $C_i(X)$ is a free $\mathbb{Z}$-module with basis the singular simplexes $\sigma : \Delta^i \to X \Rightarrow C_i(X) \otimes M \cong \bigoplus_{\sigma : \Delta^i \to X} M$. So one may think of a chain in $C_i(X) \otimes M$ as a finite linear combination with coefficients $m_j \in M$ of singular simplexes σ_j: $\sum_{j=1}^{k} \sigma_j \otimes m_j$.

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28 Feb

Def (**Eilenberg–Steenrod Axioms**, from Alg Top I)

A **homology theory** is the following.

Data: For all $n \in \mathbb{Z}$:

* Functors h_n from Cat of pairs of spaces $\to \mathbb{Z}$-Mod.
* Natural Homomorphisms $\partial : h_{n+1}(X, A) \to h_n(A) := h_n(A, \emptyset)$

$$\begin{tikzcd} h_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & h_n(A) \arrow[d, "f_*"] \\ h_{n+1}(Y,B) \arrow[r, "\partial"] & h_n(B) \end{tikzcd}$$

Commutes for all Cont. $f : (X, A) \to (Y, B)$

Axioms:

(1) $f \simeq g \Rightarrow f_* = g_*$ (Homotopy)

(2) $\overline{U} \subseteq A^\circ$, inclusion $i : (X \setminus U, A \setminus U) \to (X, A) \Rightarrow i_*$ iso (Excision)

(3) $h_n(\text{one point space}) = 0$ for $n \neq 0$ (Dimension)

(4) For inclusions $i_\alpha : X_\alpha \to \coprod_\alpha X_\alpha$,

$$\bigoplus h_n(X_\alpha) \xrightarrow{\sum_\alpha (i_\alpha)_*} h_n\left(\coprod_\alpha X_\alpha\right) \text{ is an iso.}$$ (Additivity)

(5) There are long exact sequences (Exactness)

$$\cdots \to h_n(A) \xrightarrow{incl_*} h_n(X) \xrightarrow{incl_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$$

A more precise Goal **Thm 5** $H_n(\cdot\,; M)$ is a homology theory.

Prop 6 $\mathcal{F}: R\text{-Mod} \longrightarrow \mathcal{E}$ an additive functor.

(1) An additive functor $Ch(R\text{-Mod}) \longrightarrow Ch(\mathcal{E})$, which we also denote by $\mathcal{F}$, is given by sending a chain complex C,

$$\mathcal{F}(C) = \quad \dots \longrightarrow \mathcal{F}C_2 \xrightarrow{\mathcal{F}d_2} \mathcal{F}C_1 \xrightarrow{\mathcal{F}d_1} \mathcal{F}C_0 \longrightarrow 0$$

and a chain map $f: C \longrightarrow C'$ to $\mathcal{F}(f)$ with $\mathcal{F}(f)_i = \mathcal{F}(f_i)$.

(2) If $f, g: C \longrightarrow C'$ are homotopic, then so are $\mathcal{F}(f)$ and $\mathcal{F}(g)$.

(3) $f: C \longrightarrow C'$ a homotopy equivalence $\Rightarrow$ so is $\mathcal{F}f$.

Proof (1) $\mathcal{F}d_1 \circ \mathcal{F}d_2 = \mathcal{F}(d_1 \circ d_2) = \mathcal{F}0 = 0$ ✓

$$\begin{tikzcd} C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{i-1} \arrow[d, "f_{i-1}"] \\ C'_i \arrow[r, "d'_i"'] & C'_{i-1} \end{tikzcd} \quad \overset{\mathcal{F}}{\longmapsto} \quad \begin{tikzcd} \mathcal{F}C_i \arrow[r, "\mathcal{F}d_i"] \arrow[d, "\mathcal{F}f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}C_{i-1} \arrow[d, "\mathcal{F}f_{i-1}"] \\ \mathcal{F}C'_i \arrow[r, "\mathcal{F}d'_i"'] & \mathcal{F}C'_{i-1} \end{tikzcd} ✓$$

Check that $\mathcal{F}$ is an additive functor.

(2) $f \simeq g \Rightarrow \exists$ homotopy $h: C \longrightarrow C'$, ie $h_i: C_i \longrightarrow C'_{i+1}$,

$$\begin{tikzcd} C_{i+1} \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_i \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & C_{i-1} \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] \\ C'_{i+1} \arrow[r, "d'"'] & C'_i \arrow[r, "d'"'] & C'_{i-1} \end{tikzcd}$$

with

$$hd + d'h = f - g$$

$\Rightarrow \mathcal{F}h: \mathcal{F}C \longrightarrow \mathcal{F}C'$ homotopy and $\mathcal{F}h\,\mathcal{F}d + \mathcal{F}d'\,\mathcal{F}h = \mathcal{F}f - \mathcal{F}g$.

(3) $g: C' \longrightarrow C$ and $f \circ g \simeq id_{C'}$, $g \circ f \simeq id_C$ $\Rightarrow$

$\mathcal{F}(f) \circ \mathcal{F}(g) \simeq id_{\mathcal{F}(C')}$, $\qquad \mathcal{F}(g) \circ \mathcal{F}(f) \simeq id_{\mathcal{F}(C)}$

□

Corollary 7 (apply Prop 6 to $\mathcal{F} = - \otimes M$)

(1) $C(X,A) \otimes M$ is a chain complex (that was Prop 1)

(2) Cont. $f : (X,A) \to (Y,B)$ induce chain maps

$$f_c \otimes \mathrm{id}_M : C(X,A) \otimes M \longrightarrow C(Y,B) \otimes M .$$

(3) $f \simeq g \Rightarrow f_c \otimes M \simeq g_c \otimes M$.

(4) $f_c \otimes M$ induces $f_* : H_n(X,A;M) \to H_n(Y,B;M)$

Notation We'll write f_c for $f_c \otimes \mathrm{id}_M$.

Overview of functors

$$\begin{tikzcd}
(X,A) \arrow[d, mapsto, "\substack{\text{Construction} \\ \text{in AlgTop I}}"'] & \\
C(X,A) \arrow[r, mapsto, "- \otimes M"] \arrow[d, mapsto, "\text{homology}"'] & C(X,A) \otimes M \arrow[d, mapsto, "\text{homology}"] \\
H_n(X,A) & H_n(X,A;M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Cont. } f \arrow[d, mapsto] & \\
\text{Chain maps } f_c \arrow[r, mapsto] \arrow[d, mapsto] & f_c \arrow[d, mapsto] \\
\text{homom } f_* & f_*
\end{tikzcd}$$

Rmk 8 For a commutative ring S, $C(X,A) \otimes S$ is a chain complex over S, $H_i(X,A;S)$ is an S-module, and f_c and f_* are S-linear. Particularly useful for S a field!

We have constructed half of the data to show $H_n(- ; M)$ is a homology theory, and we have proved axiom (1) (Homotopy)

Proof of Axiom (2) (Excision) $i_c : C(X \setminus U, A \setminus U) \to C(X, A)$ is a homotopy equivalence (Alg Top I).
$- \otimes M : Ch(R\text{-Mod}) \to Ch(R\text{-Mod})$ preserves homotopy equiv. (by Prop 5(3)).

$\Rightarrow$ $i_c \otimes M$ is a hom. equiv.
$\Rightarrow$ $i_* : H_n(X \setminus U, A \setminus U ; M) \to H_n(X, A ; M)$ is an iso.

$\square$

Proof of Axiom (3) (Dimension) For X the one-point space,

$$C(X) \cong \quad \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\Rightarrow C(X) \otimes M \cong \quad \cdots \xrightarrow{id_M} M \xrightarrow{0} M \xrightarrow{id_M} M \xrightarrow{0} M \to 0$$

$$\Rightarrow H_n(X ; M) \cong \begin{cases} M & n = 0 \\ 0 & \text{else} \end{cases}$$

$\square$

Proof of Axiom (4) (Additivity) $\bigoplus_\alpha C(X_\alpha) \xrightarrow{\sum (i_\alpha)_c} C(X)$ is a homotopy equiv. (Alg Top I) $\Rightarrow$ so is $\left(\bigoplus C(X_\alpha)\right) \otimes M \xrightarrow{(\sum (i_\alpha)_c) \otimes id_M} C(X) \otimes M$,

which is isomorphic to $\bigoplus (C(X_\alpha) \otimes M) \xrightarrow{\sum (i_\alpha)_c \otimes id_M} C(X) \otimes M$ $\square$

Construction of connecting maps ∂ and Proof of Axiom (5) (Exactness)

$0 \to C(A) \xrightarrow{\text{incl}} C(X) \xrightarrow{\text{incl}} C(X,A) \to 0$ is a SES of chain complexes of free abelian groups $\Rightarrow$

$0 \to C(A)\otimes M \xrightarrow{\text{incl}} C(X)\otimes M \xrightarrow{\text{incl}} C(X,A)\otimes M \to 0$

is also exact! (Exercise)

This concludes the proof, using:

Lemma 8 (Alg Top I) If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ is a SES of chain complexes over a ring, then there is a LES in homology:

$$\dots \to H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \to \dots$$

Moreover, the ∂ may be chosen naturally, which means:

If
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] \arrow[d, "\alpha"'] & D \arrow[r, "g"] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0 \end{tikzcd}$$
is commutative with exact rows

then
$$\begin{tikzcd} H_n(E) \arrow[r, "\partial"] \arrow[d, "\gamma_*"'] & H_{n-1}(C) \arrow[d, "\alpha_*"] \\ H_n(E') \arrow[r, "\partial"] & H_{n-1}(C') \end{tikzcd}$$
commutes. $\square$

Useful theorems for homology with $\mathbb{Z}$-coefficients may now be generalized to arbitrary coefficients M in one of the following ways:

* Deduce from Eilenberg–Steenrod axioms
* Deduce from the $\mathbb{Z}$-version
* Prove in the same way as for $\mathbb{Z}$

Prop 9 $H_0(X;M) \cong \bigoplus_{Z\in\pi_0(X)} \underbrace{\{[\sigma_Z\otimes m] \mid m\in M\}}_{\cong M}$, where one chooses $\sigma_Z : \underset{\{*\}}{\overset{}{\Delta^0}} \to X$, $\sigma(*)\in Z$ for each path-connected comp. $Z\in\pi_0(X)$.

Theorem 10 (Mayer-Vietoris) If $A, B \subseteq X$ with $A^\circ \cup B^\circ = X$, then there is a LES

$$\cdots \to H_n(A \cap B; M) \xrightarrow{\binom{incl_*}{incl_*}} H_n(A;M) \oplus H_n(B;M) \xrightarrow{(incl_* \ -incl_*)} H_n(X;M) \to H_{n-1}(A\cap B; M) \to \cdots$$

Theorem 11 If (X, A) is a good pair (ie $A \subseteq X$ is closed and a strong deformation retract of X), then the projection map $p: X \to X/A$ induces isos

$$p_*: H_n(X, A; M) \to H_n(X/A, A/A; M) \cong \tilde{H}_n(X/A; M)$$

Remark 12 Reduced homology groups $\tilde{H}_n(X;M)$ may be defined as over $\mathbb{Z}$ coefficients for $X \neq \emptyset$. One has

$$\tilde{H}_n(X;M) \cong H_n(X, \{x_0\}; M) \underset{\text{if } n>0}{\cong} H_n(X)$$

and $H_0(X;M) \cong M \oplus \tilde{H}_0(X;M)$.

Def (AlgTopI) X a CW-complex with cells e^n_α. Let
$C_n^{CW}(X)$ = free abelian group with basis e^n_α and
$d: C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ given by $d e^n_\alpha = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$,

where $d_{\alpha\beta} \in \mathbb{Z}$ is the degree of

$$S^{n-1} \xrightarrow[\text{map of } e^n_\alpha]{\text{attaching}} X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta) \cong S^{n-1}$$

$(n-1)$-skeleton of $X = \bigcup_{\substack{k\leq n \\ \alpha}} e^k_\alpha$

$C^{CW}(X)$ is the cellular chain complex of X and
$H_n^{CW}(X) := H_n(C^{CW}(X))$ the cellular homology of X.

Theorem 13 $H_n^{CW}(X;M) := H_n(C^{CW}(X) \otimes M) \cong H_n(X;M)$

(3) Calculations & the Theorem of Borsuk-Ulam

Prop 1 For all $k \geq 0$, $\tilde{H}_n(S^k; M) \cong M$ if $n = k$, trivial otherwise.

Three ways to prove it (1) S^k has a CW structure with one 0-cell, one k-cell.

(2) Mayer-Vietoris with $A = S^k \setminus e_1$, $B = S^k \setminus -e_1$

(3) LES of the good pair $(D^k, \partial D^k)$ $\square$

Def Real Projective k-space $\mathbb{R}P^k := S^k / x \sim -x$

Rmk 2 * $\mathbb{R}P^k \cong (\mathbb{R}P^{k+1} \setminus \vec{0}) / x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus 0$

* $\mathbb{R}P^0$ = one point space, $\mathbb{R}P^1 \cong S^1$

* Alg Top I: $$H_n(\mathbb{R}P^k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & 1 \leq n \leq k-1,\ n \text{ odd} \\ 0 & 1 \leq n \leq k-1,\ n \text{ even} \\ \mathbb{Z} & n = k \text{ odd} \\ 0 & n = k \text{ even} \\ 0 & k+1 \leq n \end{cases}$$

Prop 3 $H_n(\mathbb{R}P^k; \mathbb{Z}/2) \cong \mathbb{Z}/2$ if $0 \leq n \leq k$ and 0 otherwise.

Prop 4 Let $f: Y \longrightarrow X$ be a twofold covering. Then there is a LES

$$\dots \longrightarrow H_n(X; \mathbb{Z}/2) \to H_n(Y; \mathbb{Z}/2) \xrightarrow{f_*} H_n(X; \mathbb{Z}/2) \to H_{n-1}(X; \mathbb{Z}/2) \to \dots$$

(a special case of the Gysin LES)

Proof Recall that: a cont. map $\sigma: Z \longrightarrow X$ on a contractible space Z has exactly two lifts $\tilde{\sigma}_1, \tilde{\sigma}_2 : Z \longrightarrow Y$. Here, a lift is a map $\tilde{\sigma}: Z \to Y$ so that

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & X \end{tikzcd}$$

commutes.

Define the so-called transfer homomorphism $T: C_n(X) \to C_n(Y)$ by $T(\sigma: \Delta^n \to X) = \tilde{\sigma}_1 + \tilde{\sigma}_2$. Check that T is a chain map. We'll show that the short sequence of complexes

$$0 \to C(X) \otimes \mathbb{Z}/2 \xrightarrow{T} C(Y) \otimes \mathbb{Z}/2 \xrightarrow{f_c} C(X) \otimes \mathbb{Z}/2 \to 0$$

is exact. This induces the desired LES in homology (Lemma 2.9).

* $\underline{f_c \text{ surjective}}$ Lifts exist.
* $\underline{T \text{ is injective}}$. For a sing simplex $\tau: \Delta^n \to X$, let $p_\tau: C(X) \otimes \mathbb{Z}/2 \to \mathbb{Z}/2$ be the projection $\sum_\sigma \sigma \otimes \lambda_\sigma \mapsto \lambda_\tau$.

  $c = \sum_\sigma \sigma \otimes \lambda_\sigma \neq 0 \Rightarrow \exists\ \tau$ with $\lambda_\tau = 1$ for some τ

  $\Rightarrow \lambda_{\tilde{\tau}}(T(c)) = 1$ for $\tilde{\tau}$ a lift of $\tau \Rightarrow T(c) \neq 0$.
* $\underline{\operatorname{im}(T) = \ker f_c}$. $f_c(c = \sum_\sigma \sigma \otimes \lambda_\sigma) = 0$

  $\iff p_\tau(f_c(c)) = 0 \quad \forall \tau: \Delta^n \to X$.

  Since $p_\tau(f_c(c)) = p_{\tilde{\tau}_1}(c) + p_{\tilde{\tau}_2}(c)$, it follows that

  $$f_c(c) = 0 \iff c = \sum_{\tau: \Delta^n \to X} \lambda_\tau(\tilde{\tau}_1 + \tilde{\tau}_2) = T\Big(\sum_\tau \lambda_\tau \tau\Big)$$

  $$\iff c \in \operatorname{im}(T).$$

$\square$

Prop 4 Let $f: Y \to X$ be a twofold covering. Then there is a LES

$$\cdots \to H_n(X;\mathbb{Z}/2) \to H_n(Y;\mathbb{Z}/2) \xrightarrow{f_*} H_n(X;\mathbb{Z}/2) \to H_{n-1}(X;\mathbb{Z}/2) \to \cdots$$

(a special case of the Gysin LES)

Today For the remainder of ③: $H_n(X,A)$ means $H_n(X,A;\mathbb{Z}/2)$

Prop 3 $H_n(\mathbb{R}P^k) \cong \mathbb{Z}/2$ if $0 \le n \le k$ and 0 otherwise.

Proof We already know this for $n = 0,1$. So assume $n \ge 2$.

For the covering $f: S^n \to \mathbb{R}P^n$, the Gysin LES breaks into pieces:

$$\boxed{0 \to H_1(\mathbb{R}P^n) \xrightarrow{\partial} H_0(\mathbb{R}P^n) \xrightarrow{T_*} H_0(S^n) \xrightarrow{f_*} H_0(\mathbb{R}P^n) \to 0}$$

All homology groups are $\mathbb{Z}/2$ - vector spaces (by Rmk 2 8).

f_* surjective and $H_0(S^n) \Rightarrow H_0(\mathbb{R}P^n) \cong \mathbb{Z}/2$ or 0.

Exactness at $H_0(S^n) \Rightarrow H_0(\mathbb{R}P^n) \cong \mathbb{Z}/2 \Rightarrow f_* = 1 \Rightarrow T_* = 0$

$\Rightarrow H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$.

$$\boxed{0 \to H_k(\mathbb{R}P^n) \xrightarrow{\partial} H_{k-1}(\mathbb{R}P^n) \to 0} \quad \text{if } k \notin \{0,1,n,n+1\}$$

So, $H_k(\mathbb{R}P^n) \cong H_{k-1}(\mathbb{R}P^n) \Rightarrow H_k(\mathbb{R}P^n) \cong \mathbb{Z}/2$ for $k \le n-1$

by induction.

$$\boxed{0 \to H_{n+1}(\mathbb{R}P^n) \xrightarrow{\partial} H_n(\mathbb{R}P^n) \xrightarrow{T_*} \underbrace{H_n(S^n)}_{\mathbb{Z}/2} \xrightarrow{f_*} H_n(\mathbb{R}P^n) \xrightarrow{\partial} \underbrace{H_{n-1}(\mathbb{R}P^n)}_{\mathbb{Z}/2} \to 0}$$

Since $\mathbb{R}P^n$ has a CW-structure without k-cells for $k \ge n+1$

$\Rightarrow H_k(\mathbb{R}P^n) = 0$ for $k \ge n+1$.

$\Rightarrow H_n(\mathbb{R}P^n)$ surjects onto $\mathbb{Z}/2$, and injects into $\mathbb{Z}/2$

$\Rightarrow H_n(\mathbb{R}P^n) \cong \mathbb{Z}/2$. $\square$

Prop 5 The Gysin sequence from Prop 4 is natural, ie if

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[d, "\beta"] \\ Y' \arrow[r, "f'"'] & X' \end{tikzcd}$$

commutes and f, f' are twofold coverings, then

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X) \arrow[r, "T_*"] \arrow[d, "\beta_*"] & H_n(Y) \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & H_n(X) \arrow[r, "\partial"] \arrow[d, "\beta_*"] & H_{n-1}(X) \arrow[r] \arrow[d, "\beta_*"] & \cdots \\ \cdots \arrow[r] & H_n(X') \arrow[r, "T_*"'] & H_n(Y') \arrow[r, "f'_*"'] & H_n(X') \arrow[r, "\partial"'] & H_{n-1}(X') \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Proof Check that

$$\begin{tikzcd} 0 \arrow[r] & C_n(X)\otimes \mathbb{Z}/2 \arrow[r, "T"] \arrow[d, "\beta_c"'] & C_n(Y)\otimes \mathbb{Z}/2 \arrow[r, "f_c"] \arrow[d, "\alpha_c"'] & C_n(X)\otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "\beta_c"'] & 0 \\ 0 \arrow[r] & C_n(X')\otimes \mathbb{Z}/2 \arrow[r, "T"'] & C_n(Y')\otimes \mathbb{Z}/2 \arrow[r, "f'_c"'] & C_n(X')\otimes \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

commutes, then use Lemma 2.8. $\square$

Borsuk-Ulam Theorem $f: S^n \to \mathbb{R}^n$ continuous $\Rightarrow$
$\exists x \in S^n: f(x) = f(-x)$.

Proof If no such x exists, let $g: S^n \to S^{n-1}$,

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}.$$

Then $g(-x) = -g(x)$.

This contradicts the following theorem. $\square$

Theorem 6 There is a cont. map $g: S^n \to S^m$ with and $g(-x) = -g(x)$ $\iff$ $n \leq m$.

Proof If $n \leq m$, the embedding $i: (x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_{n+1}, 0, \dots 0)$ satisfies $i(-x) = -i(x)$.

For the other direction, assume $n > m \geq 1$ and

Let such a g be given. If $p_n(x) = p_n(y)$, then $p_m \circ g(x) = p_m \circ g(y)$.

Because the covering p_n is a quotient map, there is $h : \mathbb{RP}^n \to \mathbb{RP}^m$ s.t.

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "p_n"'] \arrow[dr, "p_m \circ g"] & S^m \arrow[d, "p_m"] \\
\mathbb{RP}^n \arrow[r, "h"'] & \mathbb{RP}^m
\end{tikzcd}$$

commutes.

Now, apply Prop 5 (naturality of the Gysin sequence) to the pieces of the Gysin LES (see proof of Prop 3):

$$\begin{tikzcd}
0 \arrow[r] & H_k(\mathbb{RP}^n) \arrow[r, "\text{iso}"] \arrow[d, "h_{*,k}"] & H_{k-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, "h_{*,k-1}"] & 0 \\
0 \arrow[r] & H_k(\mathbb{RP}^m) \arrow[r, "\text{iso}"'] & H_{k-1}(\mathbb{RP}^m) \arrow[r] & 0
\end{tikzcd}$$

commute for $1 \leq k \leq m-1$. Also, $h_{*,0}$ iso because $\mathbb{RP}^n, \mathbb{RP}^m$ path-connected $\Rightarrow h_{*,1}$ iso $\Rightarrow h_{*,2}$ iso $\Rightarrow \dots \Rightarrow h_{*,m-1}$ iso.

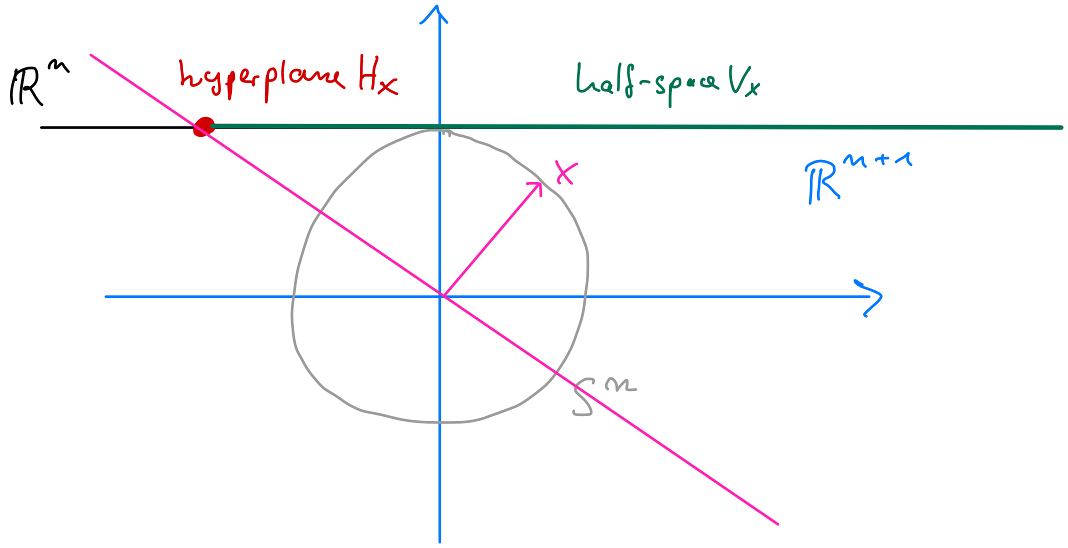
$$\begin{tikzcd}
{} \arrow[r, "\text{iso}"] & \overset{\mathbb{Z}/2}{H_m(\mathbb{RP}^n)} \arrow[r, "0"] \arrow[d, "\text{iso}"] \arrow[dr, phantom, "\text{does not commute!}"] & \overset{0}{H_m(S^n)} \arrow[r, "0"] \arrow[d] & \overset{\mathbb{Z}/2}{H_m(\mathbb{RP}^n)} \arrow[r, "\text{iso}"] \arrow[d, "\text{iso}"] & \overset{\mathbb{Z}/2}{H_{m-1}(\mathbb{RP}^n)} \arrow[r] \arrow[d, "\text{iso}"] & 0 \\
0 \arrow[r] & \underset{\mathbb{Z}/2}{H_m(\mathbb{RP}^m)} \arrow[r, "\text{iso}"'] & \underset{\mathbb{Z}/2}{H_m(S^m)} \arrow[r, "0"'] & \underset{\mathbb{Z}/2}{H_m(\mathbb{RP}^m)} \arrow[r, "\text{iso}"'] & \underset{\mathbb{Z}/2}{H_{m-1}(\mathbb{RP}^m)} \arrow[r] & 0
\end{tikzcd}$$

Contradiction! $\square$

The Ham Sandwich Theorem $A_1, \dots, A_n \subseteq \mathbb{R}^n$ Lebesgue-measurable & bounded

$\Rightarrow \exists$ hyperplane in $\mathbb{R}^n$ cutting each A_i in half by volume.

Proof Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$.



For $x \in S^n$, let
$$H_x = \mathbb{R}^n \times \{1\} \cap \{ y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle = 0 \}$$
$$V_x = \mathbb{R}^n \times \{1\} \cap \{ y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle \geq 0 \}$$

Let $f : S^n \to \mathbb{R}^n$, $f_i(x) = \mathrm{vol}(V_x \cap A_i)$.

f is continuous since the A_i are bounded.

Borsuk-Ulam $\Rightarrow \exists\, x \in S^n : f(x) = f(-x)$

$\Rightarrow \mathrm{vol}(V_x \cap A_i) = \mathrm{vol}(V_{-x} \cap A_i) = \mathrm{vol}(A_i \setminus V_x)$

$\Rightarrow H_x$ cuts all A_i in half.

$\square$

4 The Universal Coefficient Theorem for Homology

8 March

The splitting Lemma For a SES $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ of abelian groups, the following are equivalent:

(1) There is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "id_M"'] & N \arrow[r, "g"] \arrow[d, "iso"] & P \arrow[r] \arrow[d, "id_P"] & 0 \\ 0 \arrow[r] & M \arrow[r, "incl"'] & M \oplus P \arrow[r, "proj"'] & P \arrow[r] & 0 \end{tikzcd}$$

(2) $\exists\, i : P \to N$ with $g \circ i = id_P$.

(3) $\exists\, r : N \to M$ with $r \circ f = id_M$

SES satisfying these conditions are called **split**.

UCT for Homology Let C be a chain complex of free abelian groups. Let M be an abelian group.

(1) For all n, there is a split SES of abelian groups:

$$0 \to H_n(C) \otimes M \xrightarrow{[x]\otimes m \mapsto [x \otimes m]} H_n(C; M) \to \mathrm{Tor}(H_{n-1}(C), M) \to 0$$

(2) This SES is natural, ie for a chain map $f : C \to C'$

$$\begin{tikzcd} 0 \arrow[r] & H_n(C) \otimes M \arrow[r] \arrow[d, "f_* \otimes id_M"] & H_n(C; M) \arrow[r] \arrow[d, "f_*"] & \mathrm{Tor}(H_{n-1}(C), M) \arrow[r] \arrow[d, "\mathrm{Tor}(f_*, id_M)"] & 0 \\ 0 \arrow[r] & H_n(C') \otimes M \arrow[r] & H_n(C'; M) \arrow[r] & \mathrm{Tor}(H_{n-1}(C'), M) \arrow[r] & 0 \end{tikzcd}$$

commutes.

(3) There is no natural choice of splitting maps
$\longrightarrow$ Exercise 2.4

Correction 12 March
In the lecture it was erroneously claimed that "or" suffices here

Remark 1 $\mathrm{Tor}(N, M)$ will be defined for all abelian groups N, M.

We will show that ~~for~~ if M and N are finitely generated, then

$\mathrm{Tor}(N, M) \cong T(N) \otimes T(M)$, where

$T(N) = \{ x \in N \mid \exists\, \lambda \in \mathbb{Z} \setminus \{0\} : \lambda x = 0 \}$ is the torsion subgroup of N.

Remark The UCT implies that homology with any coefficients can be read off homology with $\mathbb{Z}$ coefficients, i.e. $\mathbb{Z}$ coefficients are "universal". However, for a cont. map f, f_* on $H(-;M)$ is in general <u>not</u> determined by f_* on $H(-;\mathbb{Z})$.
→ Exercise 2.4

Example 2 For $\mathbb{R}P^3$, $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}/2$, $H_2 \cong 0$, $H_3 = \mathbb{Z}$

UCT for $M = \mathbb{Z}/2$:

$$0 \to \underbrace{H_1(\mathbb{R}P^3)\otimes \mathbb{Z}/2}_{\mathbb{Z}/2} \to \underbrace{H_1(\mathbb{R}P^3;\mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\mathrm{Tor}(\overbrace{H_0(\mathbb{R}P^3)}^{\mathbb{Z}},\mathbb{Z}/2)}_{0} \to 0$$

$$0 \to \underbrace{H_2(\mathbb{R}P^3)\otimes \mathbb{Z}/2}_{0} \to \underbrace{H_2(\mathbb{R}P^3;\mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\mathrm{Tor}(\overbrace{H_1(\mathbb{R}P^3)}^{\mathbb{Z}/2},\mathbb{Z}/2)}_{\mathbb{Z}/2} \to 0$$

Reminder M finitely generated abelian group $\Rightarrow$

$$M = M^a \oplus \bigoplus_{\substack{p \text{ prime} \\ r\geq 1}} (\mathbb{Z}/_{p^r})^{b_{p,r}} \quad \text{with } a, b_{p,r} \text{ uniquely determined.}$$

a is called the rank of M, written $\mathrm{rk}\,M$ or $\mathrm{rank}\,M$.

Prop 3 Assume $\bigoplus_n H_n(X)$ is finitely generated. Let $\mathbb{F}$ be a field of characteristic p.

$$\dim_{\mathbb{F}} H_n(X;\mathbb{F}) = \begin{cases} \mathrm{rank}\, H_n(X) & \text{if } p = 0 \\ \mathrm{rank}\, H_n(X) \\ \quad + \#\ \mathbb{Z}/p^r\text{-summands of } H_n(X) & \text{else} \\ \quad + \#\ \mathbb{Z}/p^r\text{-summands of } H_{n-1}(X) \end{cases}$$

Proof UCT $\Rightarrow$ $H_n(X;\mathbb{F}) \cong H_n(X)\otimes\mathbb{F} \oplus \underbrace{\mathrm{Tor}(H_{n-1}(X),\mathbb{F})}$

$\cong T(H_{n-1}(X)) \otimes T(\mathbb{F})$ by Remark 1

<u>Correction 12 March</u>
The Proposition is true, but the proof doesn't work in general since $\mathbb{F}$ need not be finitely generated. We'll need to understand Tor better first to prove Prop 3

Now use $T(\mathbb{F}) = \begin{cases} 0 & \text{if } p=0 \\ \mathbb{F} & \text{else} \end{cases}$

and $\mathbb{Z}/m \otimes \mathbb{F} \cong \mathbb{F}/m \cong \begin{cases} 0 & p \mid m \\ \mathbb{F} & \text{else} \end{cases}$ $\square$

Prop 4 Let X be a space s.t. $H_n(X) \cong 0$ for sufficiently large n, and $H_n(X)$ finitely generated for all n. Then

$$\sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{F}}(H_n(X;\mathbb{F})) \in \mathbb{Z}$$

does not depend on the choice of a field $\mathbb{F}$. This integer is called the Euler characteristic of X, written $\chi(X)$.

Proof Note that (# $\mathbb{Z}/p^r$-summands of $H_n(X)$) appears as summand in $\dim H_n(X;\mathbb{F})$ <u>and</u> in $\dim H_{n+1}(X;\mathbb{F})$. So, this cancels in χ due to opposite signs. $\square$

To prove the UCT, we need a fundamental tool of homological algebra. Let R be a commutative ring.

Def A free resolution of an R-Module M is a LES

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where the F_i are free R-Modules.

Last time

To prove the UCT, we need a fundamental tool of homological algebra. Let R be a commutative ring.

Def A **free resolution** $\mathcal{F}$ of an R-Module M is a LES

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where the F_i are free R-Modules.

Today **13 March**

Note that $\cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$ is a chain complex. It is called **deleted resolution**, denoted $\mathcal{F}^M$, with $H_0(\mathcal{F}^M) \cong M$, $H_n(\mathcal{F}^M) \cong 0$ for $n \neq 0$. Understanding $H_n(\mathcal{F}^M; N)$ is a special case of understanding $H_n(C; N)$ for all complexes!

Ex For $R = \mathbb{Z}$:

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & M & & \\
\cdots \to 0 & \to & 0 & \to & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \longrightarrow & & & \mathbb{Z}/3 & \to & 0 \\
 & & & & \cdots & & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 \\
 & & \cdots 0 & \to & \mathbb{Z} & \xrightarrow{\binom{1}{-1}} & \mathbb{Z}^2 & \xrightarrow{(1\ \ 1)} & & & \mathbb{Z} & \to & 0 \\
 & & & & & & & & ? & \to & \mathbb{Q} & \to & 0
\end{array}$$

Prop 5 Every module has a free resolution.

Lemma 6 For every module M there exists a free module F with a surjection $p: F \to M$.

Proof $F := \bigoplus_{x \in M} R_x$ with $R_x \cong R$. F is free (with basis indexed by M) and $p: F \to M$, $R_x \ni 1 \mapsto x$ is surjective. $\square$

Proof of Prop 5 Pick $d_0 : F_0 \to M$ with d_0 surjective, F_0 free.
Pick $d_1' : F_1 \to \ker d_0$ with d_1' surjective, F_1 free and let
$d_1 : F_1 \to F_0$, $d_1 = (\ker d_0 \hookrightarrow F_0) \circ d_1'$.
Pick $d_2' : F_2 \to \ker d_1$ with d_2' surjective, F_2 free ... etc. □

Thm 7 Every subgroup of a free abelian group is free abelian.
Proof using Zorn's Lemma (see eg Lang "Algebra" Appendix 2 §2)

Prop 8 For $R = \mathbb{Z}$: Every abelian group M has a free resolution of length 1, ie $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$

Proof Pick $d_0 : F_0 \to M$ with d_0 surjective, F_0 free. By Thm ,
$\ker d_0$ is free. So let $F_1 = \ker d_0$, and d_1 the inclusion. □

Prop 9 ("Comparison Thm", "Fundamental Thm of Homological Algebra")

(1) If $f : M \to N$ is R-linear and F, G are free resolutions of M, N, then f may be extended to a chain map $\hat{f} : F^M \to G^N$, ie

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[d, dashed, "\exists \hat{f}_1"] & F_0 \arrow[r, "d_0"] \arrow[d, dashed, "\exists \hat{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & G_1 \arrow[r, "e_1"'] & G_0 \arrow[r, "e_0"'] & N \arrow[r] & 0
\end{tikzcd}$$

(2) $\hat{f}$ is unique up to homotopy.

(3) F, G free resolutions of M $\Rightarrow$ The unique chain map $F^M \to G^M$ extending id_M is a homotopy equivalence.

Proof (1)
$$\begin{tikzcd} & F_0 \arrow[d, dashed, "\exists \hat{f}_0"'] \arrow[dr, "f \circ d_0"] & \\ & G_0 \arrow[r, "e_0"'] & N \end{tikzcd}$$

Since e_0 surjective and F_0 free, there is $\hat{f}_0 : F_0 \to G_0$ making the diagram commute (proof: for each basis element b of F_0, pick $\hat{f}_0(b)$ such that $e_0(\hat{f}_0(b)) = f(d_0(b))$.

$$\begin{tikzcd} F_1 \arrow[dr, "\hat{f}_0 \circ d_1"] & \\ G_1 \arrow[r, "e_1"'] & G_0 \end{tikzcd}$$

$f(d_0(d_1(x))) = 0 \quad \forall x \quad \Rightarrow e_0(\hat{f}_0(d_1(x))) = 0 \quad \forall x$

$\Rightarrow \operatorname{im} \hat{f}_0 \circ d_1 \subseteq \ker e_0 = \operatorname{im} e_1.$

$\Rightarrow \exists \hat{f}_1 : F_1 \to G_1$ making the diagr commute etc.

(2) Let two such chain maps be given, and let g be their difference. Then:

$$\begin{tikzcd} \cdots F_2 \arrow[r, "d_2"] \arrow[d, "g_2"'] & F_1 \arrow[r, "d_1"] \arrow[d, "g_1"] \arrow[dl, "h_1"] & F_0 \arrow[r, "d_0"] \arrow[d, "g_0"] \arrow[dl, "h_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ \cdots G_2 \arrow[r, "e_2"'] & G_1 \arrow[r, "e_1"'] & G_0 \arrow[r, "e_0"'] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. $0 = 0 \circ d_0 = e_0 \circ g_0 \Rightarrow \operatorname{im} g_0 \leq \ker e_0 = \operatorname{im} e_1.$

$\Rightarrow \exists h_0$ with $e_1 \circ h_0 = g_0$

$$e_1 \circ (g_1 - h_0 \circ d_1) = e_1 \circ g_1 - g_0 \circ d_1 = 0$$

$\Rightarrow \exists h_1$ with $e_2 \circ h_1 = g_1 - h_0 \circ d_1$ etc.

(3) F, G free res. of M $\Rightarrow \exists$ chain maps $\hat{f} : F^M \to G^M$ and $\hat{g} : G^M \to F^M$ that extend $\mathrm{id}_M : M \to M$ $\Rightarrow \hat{g} \circ \hat{f} : F^M \to F^M$ and $\hat{f} \circ \hat{g} : G^M \to G^M$ extend id_M, but so do $\mathrm{id}_{F^M}, \mathrm{id}_{G^M}$

$\Rightarrow$ By uniqueness, $\hat{g} \circ \hat{f} \simeq \mathrm{id}_{F^M}, \hat{f} \circ \hat{g} \simeq \mathrm{id}_{G^M}$. $\square$

Def Let M, N be R-Modules, and $\mathcal{F}$ a free resolution of M, then $\mathrm{Tor}_n(M,N) := H_n(\mathcal{F}^M; N)$ for $n \geq 0$.

Proof that Tor does not depend on choice of $\mathcal{F}$: $\mathcal{F}, \mathcal{G}$ free res. of M $\Rightarrow \mathcal{F}^M \simeq \mathcal{G}^M \Rightarrow \mathcal{F}^M \otimes N \simeq \mathcal{G}^M \otimes N$ (Cor ② 7 (3)) $\Rightarrow$ $H_n(\mathcal{F}^M; N) \cong H_n(\mathcal{G}^M; N)$. $\square$

Remark 10 Over $R = \mathbb{Z}$, $\mathrm{Tor}_n(M,N) = 0$ $\forall n \geq 2$ since M has a free res. of length 1 (Prop 8). So we write $\mathrm{Tor}(M,N) := \mathrm{Tor}_1(M,N)$.

Lemma 11 $f: M \to N$ R-linear, P R-module $\Longrightarrow$ $(\mathrm{Coker}\, f) \otimes P \cong \mathrm{Coker}(f \otimes id_P)$. **Proof** Exercise.

Proof of the UCT (1) Constructing the SES

$$\underbrace{B_n = \mathrm{im}\, d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \ker d_n}_{n\text{-cycles}}$$

Make B_n, Z_n into chain complexes, taking 0 as differential.
There is a SES of chain complexes:

$$\begin{tikzcd}
 & & \vdots & & \\
0 \arrow[r] & Z_{n+1} \arrow[r, "incl"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d"] \arrow[d, "d"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "incl"] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd}$$

Proof of the UCT (1) Constructing the SES

15 March

$$\underbrace{B_n = \operatorname{im} d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \ker d_n}_{n\text{-cycles}}$$

Make B_n, Z_n into chain complexes, taking 0 as differential.

There is a SES of chain complexes:

$$\begin{tikzcd} & & \vdots & & \\ 0 \arrow[r] & Z_{n+1} \arrow[r, "\text{incl}"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d"] \arrow[d, "d"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "\text{incl}"] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \\ & & \vdots & & \end{tikzcd}$$

B_n free by Thm 7 $\Rightarrow$ each row splits $\Rightarrow$ tensoring with M preserves exactness (Exercise). The SES$\otimes M$ induces a LES:

$$\cdots \to B_n \otimes M \xrightarrow{r} Z_n \otimes M \longrightarrow \frac{\ker d_n \otimes \mathrm{id}_M}{\operatorname{im} d_{n+1} \otimes \mathrm{id}_M} \longrightarrow B_{n-1} \otimes M \xrightarrow{\text{incl} \otimes \mathrm{id}_M} Z_{n-1} \otimes M \to \cdots$$

$$\Rightarrow \text{SES} \quad 0 \to \underbrace{H_n(C) \otimes M}_{\substack{\cong \operatorname{coker} r \text{ by} \\ \text{Lemma 11}}} \to \overset{\substack{\ker d_n \otimes \mathrm{id}_M / \operatorname{im} d_{n+1} \otimes \mathrm{id}_M \\ \cong}}{H_n(C; M)} \to \ker \text{incl} \otimes \mathrm{id}_M \to 0$$

There is a SES

$$0 \to B_{n-1} \xrightarrow{\text{incl}} Z_{n-1} \longrightarrow H_{n-1}(C) \to 0$$

which is a free resolution of $H_{n-1}(C)$. So

$$\ker \text{incl} \otimes \mathrm{id}_M \cong \operatorname{Tor}(H_{n-1}(C), M).$$

(1) <u>The SES splits</u> $\quad C_m$ free $\Rightarrow \exists\ p_m : C_m \to Z_m$ s.t.

$\mathrm{incl} \circ p_m = \mathrm{id}_{Z_m}$. Correction 5 April $p : C \to Z$ is in general <u>not</u> a chain map! (Indeed, p chain map $\Rightarrow$ differential of C is zero). Proceed instead as follows: Let $\pi_m : Z_m \to H_m(C) = Z_m/B_m$ be the projection. Then $\pi_m \circ p_m$ is a map $C_m \to H_m(C)$, and this <u>is</u> a chain map when one considers $H_m(C)$ as complex with zero differential (since for $x \in C_m$: $d_m(x) \in B_{m-1} \subseteq Z_{m-1}$, so $p_{m-1}(d_m(x)) = d_m(x)$ and $\pi_{m-1}(p_{m-1}(d_m(x))) = [d_m(x)] = 0$).

Thus $(\pi_m \circ p_m) \otimes \mathrm{id}_M : C_m \otimes M \to H_m(C) \otimes M$ is also a chain map, inducing a map $H_m(C;M) \xrightarrow{q} H_m(C) \otimes M$ on homology. To see that q is a splitting map, check that $q([x \otimes m]) = [x] \otimes m$ for all $x \in Z_m$ and $m \in M$.

(2) <u>Naturality</u> (Sketch)

$f : C \to C'$ chain map $\Rightarrow f(Z) \subseteq Z'$, $f(B) \subseteq B'$.

So f induces a map between the SES of chain complexes

$0 \to Z_m \to C_m \to B_{m-1} \to 0$ and $0 \to Z'_m \to C'_m \to B'_{m-1} \to 0$,

also after $\otimes M$, and so also between the associated LES, and so also between the SES in the UCT.

(3) <u>Unnaturality of splitting</u>: Exercise 2.4

Prop 12 $\quad \mathrm{Tor}_0(M,N) \cong M \otimes N$

Proof $\quad \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$ deleted free res of M.

$$\Rightarrow \mathrm{Tor}_0(M,N) = \mathrm{coker}(d_1 \otimes \mathrm{id}_N) \cong \mathrm{coker}(d_1) \otimes N$$

$$= H_0(F^M) \otimes N = M \otimes N$$ $\square$

Remark 13 For $f : M \to M'$, $g : N \to N'$, one may set

$\mathrm{Tor}_m(f,g) : \mathrm{Tor}_m(M,N) \to \mathrm{Tor}_m(M',N')$ to be given by $(\hat{f} \otimes g)_*$. Fixing one argument then makes Tor_m into an additive functor $R\text{-Mod} \to R\text{-Mod}$.

Prop 14 Let A, B, C be abelian groups.

(1) B free $\Rightarrow$ $\mathrm{Tor}(A,B) \cong 0$

(2) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then

$$0 \to \mathrm{Tor}(D,A) \to \mathrm{Tor}(D,B) \to \mathrm{Tor}(D,C) \to D\otimes A \to D \otimes B \to D \otimes C \to 0$$

is exact.

(3) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

(4) B torsion-free $\Rightarrow$ $T(A,B) \cong 0$

(5) $T(A,B) \cong \mathrm{Tor}(T(A), T(B))$.

(6) $\mathrm{Tor}(\mathbb{Z}/n, A) \cong \{x \in A \mid nx = 0\}$

(7) $\mathrm{Tor}(A \oplus B, C) \cong \mathrm{Tor}(A,C) \oplus \mathrm{Tor}(B,C)$

(8) $\mathrm{Tor}(A,B) \cong T(A) \otimes T(B)$ if A and B are f.g.

Proof (1) $0 \to F_1 \to F_0 \to A \to 0$ free res of A $\Rightarrow$
$0 \to F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0$ is exact $\Rightarrow \mathrm{Tor}(A,B) \cong 0$.

(2) Pick free res $0 \to F_1 \xrightarrow{d_1} F_0 \to D \to 0$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & F_1 \otimes A \arrow[r, "id_{F_1}\otimes f"] \arrow[d, "d_1 \otimes id_A"'] & F_1 \otimes B \arrow[r, "id_{F_1}\otimes g"] \arrow[d, "d_1 \otimes id_B"'] & F_1 \otimes C \arrow[r] \arrow[d, "d_1 \otimes id_C"'] & 0 \\ 0 \arrow[r] & F_0 \otimes A \arrow[r, "id_{F_0}\otimes f"'] & F_0 \otimes B \arrow[r, "id_{F_0}\otimes g"'] & F_0 \otimes C \arrow[r] & 0 \end{tikzcd}$$

commutes and has exact rows. It is a SES of chain complexes! (Each complex made of two groups). The associated LES in homology is the desired sequence. ✓

(3) Apply (1) to a free res $0 \to F_1 \xrightarrow{d_1} F_0 \longrightarrow B \longrightarrow 0$

$\rightsquigarrow$ LES

$$0 \to \underbrace{\mathrm{Tor}(A, F_1)}_{0 \text{ because } F_1 \text{ free}} \to \underbrace{\mathrm{Tor}(A, F_0)}_{0 \text{ because } F_0 \text{ free}} \to \mathrm{Tor}(A,B) \to A \otimes F_1 \xrightarrow{id_A \otimes d_1} A \otimes F_0 \longrightarrow A \otimes B \longrightarrow 0$$

$\Rightarrow$ $\mathrm{Tor}(A,B) \cong \ker(id_A \otimes d_1) = \mathrm{Tor}(B,A)$ by def of Tor, using $A \otimes B \cong B \otimes A$. ✓

(4) Pick free res $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \longrightarrow 0$.

It's enough to show that $F_1 \otimes B \longrightarrow F_0 \otimes B$ is injective.

So let $\alpha \in F_1 \otimes B$ with $d_1 \otimes id_B(\alpha) = 0$ be given. To show: $\alpha = 0$.

<u>Claim</u> There is a f.g. subgroup $B' \subseteq B$ with $\alpha \in B'$ and $d_1 \otimes id_{B'}(\alpha) = 0$.

<u>Pf that Claim $\Rightarrow \alpha = 0$</u> B torsion free $\Rightarrow B'$ torsion free. B' torsion free and f.g. $\Rightarrow B'$ free by classification of f.g. ab. groups. We already know that tensoring with a free module is exact $\Rightarrow$ $d_1 \otimes id_{B'}$ injective $\Rightarrow \alpha = 0$. ✓

<u>Pf of Claim</u> Use construction of $\otimes$: $F_0 \otimes B \cong$ free module $U_{F_0,B}$ with basis $F_0 \times B$ modulo submodule $I_{F_0,B} \subseteq U$ generated by

$$\left.\begin{array}{l} (\lambda x + x', y) - \lambda(x,y) - (x',y) \\ (x, \lambda y + y') - \lambda(x,y) - (x,y') \end{array}\right. \qquad (*)$$

Write $\alpha = \sum_{i=1}^{m} f_i \otimes b_i$. Then $d_1 \otimes id_B(\alpha) = 0 \Leftrightarrow \sum d_1(f_i) \otimes b_i = 0$

$\Leftrightarrow \sum_{i=1}^{m} (d_1(f_i), b_i) = \sum_{j=1}^{k}$ elements of the form $(*)$ $\in I_{F_0,B}$

Let $B' \subseteq B$ be generated by $b_1, \dots, b_m$ and all elements of B appearing in the sum on the RHS. Then $\alpha \in F_1 \otimes B'$, and $d_1 \otimes id_{B'}(\alpha) = 0$ ✓

the following proofs were skipped in the lecture

(5) Apply (2) to the SES $0 \to T(B) \to B \to B/T(B) \to 0$:

$$0 \to \mathrm{Tor}(A, T(B)) \to \mathrm{Tor}(A,B) \to \underbrace{\mathrm{Tor}(A, B/T(B))}_{\substack{0 \text{ by (4) since} \\ B/T(B) \text{ torsion-free}}} \to \cdots$$

$\Rightarrow \mathrm{Tor}(A, T(B)) \cong \mathrm{Tor}(A,B)$. Now use (3) and repeat the argument. ✓

(6) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ is a free res of $\mathbb{Z}/n$.

$\Rightarrow \mathrm{Tor}(\mathbb{Z}/n, A) \cong \ker(A \xrightarrow{n} A) = \{x \in A \mid nx = 0\}$ ✓

(7) $\left.\begin{array}{l} 0 \to F_1 \to F_0 \to A \to 0 \\ 0 \to G_1 \to G_0 \to B \to 0 \end{array}\right\}$ free res.

$\Rightarrow 0 \to F_1 \oplus G_1 \to F_0 \oplus G_0 \to A \oplus B \to 0$ free res

Now
$$\begin{aligned} \mathrm{Tor}(A \oplus B, C) &\cong \ker\left((F_1 \oplus G_1) \otimes C \to (F_0 \oplus G_0) \otimes C\right) \\ &\cong \ker(F_1 \otimes C \to F_0 \otimes C) \\ &\quad \oplus \ker(G_1 \otimes C \to G_0 \otimes C) \\ &\cong \mathrm{Tor}(A, C) \oplus \mathrm{Tor}(B,C) \end{aligned}$$ ✓

(8) Using (7), (3), (1) and the classification of f.g. ab groups, it is enough to check this for $A \cong \mathbb{Z}/a$, $B \cong \mathbb{Z}/b$. This will be an Exercise on Sheet 3. ✓

$\square$

5 Cohomology

Goal Dualize the singular chain complex, ie apply $\mathrm{Hom}(-,\mathbb{Z})$ (or $\mathrm{Hom}(-,M)$ for any abelian group M) $\rightsquigarrow$ cochain complex with cohomology. Why? Cohomology ...

* ... has more structure than a homology (it is a ring!)
* ... may arise in a natural way from geometric applications

Def A cochain complex C over a commutative ring R is a collection C^n of R-modules for $n \in \mathbb{Z}$ called cochain modules, R-linear maps $d^n : C^n \to C^{n+1}$ with $d^{n+1} \circ d^n = 0$ called differentials. The n-th cohomology module of C is

n-cocycles

$$H^n(C) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}$$

n-coboundaries

A cochain map $f : C \to D$ is a collection of R-linear $f^n : C^n \to D^n$ st $f^{n+1} \circ d^n_C = d^n_D \circ f^n \ \forall n$.

$f, g : C \to D$ are homotopic, written $f \simeq g$, if $\exists$ a homotopy $h : C \to D$, ie a collection of R-linear $h^n : C^n \to D^{n-1}$, s.t. $f_n - g_n = d^{n-1}_D \circ h_n + h_{n+1} \circ d^n_C$

Remark 1 C cochain complex

$\Longleftrightarrow$ D with $D_n = C^{-n}$, $d^D_n = d^{-n}_C$ is a chain complex

Under this 1:1-correspondence, cohomology $\leftrightarrow$ homology, cochain maps $\leftrightarrow$ chain maps, homotopies $\leftrightarrow$ homotopies etc.

So everything that is true for chain complexes also holds true mutatis mutandis for cochain complexes, eg Prop 2.

Prop 2 (1) $f: C \to D$ a cochain map $\Rightarrow$

$f^*: H^n(C) \to H^n(D)$, $f^*([x]) = [f(x)]$ is a well-def. R-homom.

(2) $H^n(-)$ is an additive functor

$$\underbrace{CoCh(R)} \longrightarrow R\text{-Mod}$$

category of cochain complexes over R, cochain maps

(3) $f \simeq g \Rightarrow f^* \simeq g^*$.

No proof

Prop 3 If $F: R\text{-Mod} \to R\text{-Mod}$ is a contravariant additive functor, then $F: Ch(R) \to CoCh(R)$ is also contravariant additive:

$$\ldots C_n \xrightarrow{d_n} C_{n-1} \ldots \longmapsto \ldots F(C_n) \xleftarrow{F(d_n)} F(C_{n-1}) \ldots$$

cochain complex $F(C)$
with $F(C)^n = F(C_n)$,
$d^n_{F(C)} = F(d^{n-1}_C)$

No proof

Def X top. space, $A \subseteq X$, M an abelian group.
Then the cochain complex obtained from $C_n(X,A)$ by applying $\mathrm{Hom}(-, M)$ is called the singular cochain complex of (X,A) with coefficients in M, denoted $C^n(X,A;M)$ and its cohomology the singular cohomology of (X,A) with coefficients in M, denoted $H^n(X,A;M)$. We may drop ";M" for $M = \mathbb{Z}$.
For $f: (X,A) \to (Y,B)$ continuous, write f^c for the cochain map $C^n(Y,B;M) \to C^n(X,A;M)$, $f^c = \mathrm{Hom}(f_c, M)$, and f^* for the induced homom. $H^n(Y,B;M) \to H^n(X,A;M)$.

Ex 4 $C^0(X;M) = \mathrm{Hom}(C_0(X), M)$. Corresponds to functions $X \to M$. Let $\varphi \in C^0(X;M)$. Then $d^0(\varphi)$ sends $\sigma : \Delta^1 = [0,1] \to M$ to $\varphi(d_1(\sigma)) = \varphi(\sigma(1)) - \varphi(\sigma(0))$

So $d^0(\varphi) = 0 \Leftrightarrow \varphi(\sigma(0)) = \varphi(\sigma(1))\ \forall \sigma \Leftrightarrow \varphi$ constant on path-connected components. Hence

$$H^0(X;M) = \ker d^0 \cong \prod_{\pi_0(X)} M$$

note: for $\pi_0(X)$ infinite
$$\underset{\substack{\text{\rotatebox{90}{$\cong$}}\\ \prod_{\pi_0(X)} \mathbb{Z}}}{H^0(X;\mathbb{Z})} \ncong \underset{\substack{\text{\rotatebox{90}{$\cong$}}\\ \bigoplus_{\pi_0(X)} \mathbb{Z}}}{H_0(X;\mathbb{Z})}$$

Rmk 5 A hands-on approach to cochains:

An n-cochain $\varphi \in C^n(X;M)$ is a homom. $C_n(X) \to M$.

So n-chains correspond to functions

$$\{\text{singular } n\text{-simplices } \sigma : \Delta^n \to X\} \longrightarrow \mathbb{Z}$$

The $(n+1)$-cochain $d^n(\varphi)$ sends $\tau : \Delta^{n+1} \to X$ to $\varphi(d_{n+1}(\tau))$.

So φ is an n-cocycle $\Leftrightarrow$ φ is zero on n-boundaries $\in B_n$.

φ is an n-coboundary $\Rightarrow$ $\varphi(\sigma)$ is determined by $d_n(\sigma)$.

$\Longrightarrow$ φ is zero on n-cycles $\in Z_n$

Correction 22 April The implication "$\Leftarrow$" does not generally hold: there may be cochains φ that are zero on n-cycles, but that are not coboundaries. Indeed, this happens if φ is a cocycle, $[\varphi] \neq 0 \in H^n(X;M)$, and $ev([\varphi]) = 0$.

Thus: An n-cocycle φ induces a homom. $C_n(X)/B_n \to M$, by restriction it also induces a homom.

$$Z_n/B_n = H_n(X) \longrightarrow M.$$

For n-coboundaries φ, this homom. is zero. Thus we have a homom. called the evaluation homomorphism

$$ev : H^n(X;M) \longrightarrow \mathrm{Hom}(H_n(X), M)$$

which may be seen to be natural in both X and M.

Universal Coefficient Thm for Cohomology

Let C be a chain complex of free abelian groups and A an abelian group

(1) There is a split SES

$$0 \to \operatorname{Ext}(H_{n-1}(C), A) \to H^n(C; M) \xrightarrow[ev]{} \operatorname{Hom}(H_n(C), A) \to 0$$

to be defined!

(2) These SES are natural in C and A.

(3) The splittings cannot be chosen naturally

Def Let M, N be R-modules, and F a free res. of M. Then let

$$\mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}(F^M, N))$$

F^M unique up to hom. equiv. $\Rightarrow$ Def of Ext independent of choice of F.

As with Tor, we have:

* $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}(M,N)$.
* $\mathrm{Ext}^n_{\mathbb{Z}}(A,B) = 0$ for all $n \geq 2$, so we write $\mathrm{Ext}(A,B)$ for $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$.

For the proof of the first point, one needs:

Lemma 6 M, N, P R-modules, $f: M \to N$ R-linear

$$\Rightarrow \mathrm{Hom}(\mathrm{coker}\, f, P) \cong \ker(\mathrm{Hom}(f, P))$$

Proof $M \longrightarrow N \longrightarrow \mathrm{coker}\, f \longrightarrow 0$ exact

$\Rightarrow 0 \to \mathrm{Hom}(\mathrm{coker}\, f, P) \longrightarrow \mathrm{Hom}(N,P) \to \mathrm{Hom}(M,P)$ is exact

(same argument as in Ex Sheet 1, 2b) □

Rmk 7 * Ext is not symmetric: $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$

$\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/n) \cong 0$

(as we shall see from Prop 8)

* Ext can behave unexpectedly:

$$\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \text{uncountably-dimensional } \mathbb{Q}\text{-vector space}$$

Prop 8 For all ab groups A, B, C, the following hold:

(1) $\mathrm{Ext}(A \oplus B, C) \cong \mathrm{Ext}(A, C) \oplus \mathrm{Ext}(B, C)$

(2) $\mathrm{Ext}(A, B \oplus C) \cong \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A, C)$

(3) A free $\Rightarrow \mathrm{Ext}(A, B) \cong 0$.

(4) $\mathrm{Ext}(\mathbb{Z}/n, A) \cong A/nA$

Note this suffices to compute $\mathrm{Ext}(\text{f.g. group}, A)$.

(5) $\mathrm{Ext}(A, B) \cong T(A) \otimes B$ if A, B f.g.

Compare (4), (5) to Tor: $\mathrm{Tor}(\mathbb{Z}/n, A) \cong \{x \in A \mid nx = 0\}$

$\mathrm{Tor}(A, B) \cong T(A) \otimes T(B)$ for A, B f.g.

Proof of (4) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ free res. $\mathcal{F}$

$$\mathrm{Hom}(\mathcal{F}^{\mathbb{Z}/n}, A) = 0 \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z}, A)}_{\cong A} \xleftarrow{n} \underbrace{\mathrm{Hom}(\mathbb{Z}, A)}_{\cong A} \leftarrow 0$$

$\Rightarrow \mathrm{Ext} = H^1$ of this cochain complex $\cong A/nA$ $\square$

Rmk 9 Let R-modules M, N be given. An extension of N by M is a SES $0 \to N \to P \to M \to 0$. It is equivalent to another extension $0 \to N \to P' \to M \to 0$ if $\exists f: P \to P'$ st

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "id_N"'] & P \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "id_M"] & 0 \\
0 \arrow[r] & N \arrow[r] & P' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

commutes, Five-Lemma $\Rightarrow f$ is iso. So equivalence is an equiv. rel. One finds $\{\text{Extensions of } N \text{ by } M\}/\text{equiv} \xleftrightarrow{1:1} \mathrm{Ext}^1_R(M, N)$.

Prop 10 Assume $H_n(X,A)$ is f.g. for all n. Then

$$H^n(X,A;\mathbb{Z}) \cong \underbrace{F(H_n(X,A))}_{\text{free part}} \oplus T(H_{n-1}(X,A))$$

$F(B) := B/T(B)$

Proof UCT $\Rightarrow$

$$H^n(X,A;\mathbb{Z}) \cong \operatorname{Hom}(H_n(X,A),\mathbb{Z}) \oplus \operatorname{Ext}(H_{n-1}(X,A),\mathbb{Z})$$

$$\begin{aligned}
&\cong \operatorname{Hom}(F(H_n(X,A)),\mathbb{Z}) && \cong F(H_n(X,A))\\
&\oplus \operatorname{Hom}(T(H_n(X,A)),\mathbb{Z}) && \cong 0\\
&\oplus \operatorname{Ext}(F(H_{n-1}(X,A)),\mathbb{Z}) && \cong 0\\
&\oplus \operatorname{Ext}(T(H_{n-1}(X,A)),\mathbb{Z}) && \cong T(H_{n-1}(X,A))
\end{aligned}$$

$\square$

Def The cellular cochain complex $C^\bullet_{CW}(X)$ of a CW-complex X is $\operatorname{Hom}(C^{CW}_\bullet(X), M)$. Its cohomology $H^n_{CW}(X;M)$ is the n-th cellular cohomology group.

Thm 11 $H^n_{CW}(X;M) \cong H^n(X;M)$.

Example 12 $C^{CW}_\bullet(\mathbb{R}P^2) = 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

$H^{CW}_0 \cong \mathbb{Z}$, $H^{CW}_1 \cong \mathbb{Z}/2$, $H^{CW}_2 = 0$

Hands-on Trick: C a chain complex of f.g. free ab. groups with a chosen basis, then

$$\underset{\text{wrt to the basis}}{(\text{Matrix of } d_n)^T} = \underset{\text{wrt the dual basis}}{\text{Matrix of } \operatorname{Hom}(d_n,\mathbb{Z})}$$

$\Rightarrow C^\bullet_{CW}(\mathbb{R}P^2;\mathbb{Z}) = 0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$

and $H^0_{CW} \cong \mathbb{Z}$, $H^1_{CW} \cong 0$, $H^2_{CW} \cong \mathbb{Z}/2$

Proof of UCT (1)

There is a SES of chain complexes:

$$\begin{tikzcd} & \vdots & & & \\ 0 \arrow[r] & Z_{n+1} \arrow[r, "incl"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_{n+1}"'] & B_n \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "incl"] & C_n \arrow[r, "d_n"] & B_{n-1} \arrow[r] & 0 \\ & \vdots & & & \end{tikzcd}$$

B_n free by Thm 4.7 $\Rightarrow$ each row splits $\Rightarrow$ SES of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(B_{n-1},M) \arrow[r, "d^{n-1}"] \arrow[d, "0"'] & \mathrm{Hom}(C_n,M) \arrow[r, "incl^*"] \arrow[d, "d^n"'] & \mathrm{Hom}(Z_n,M) \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(B_n,M) \arrow[r, "d^n"] & \mathrm{Hom}(C_{n+1},M) \arrow[r, "incl^*"] & \mathrm{Hom}(Z_{n+1},M) \arrow[r] & 0 \end{tikzcd}$$

This induces a LES

$$\ldots \to \mathrm{Hom}(Z_{n-1},M)$$
$$\xrightarrow{\partial^{n-1}} \mathrm{Hom}(B_{n-1},M) \to H^n(C;M) \to \mathrm{Hom}(Z_n,M)$$
$$\xrightarrow{\partial^n} \mathrm{Hom}(B_n,M) \to \ldots$$

Check that $\partial^i = \mathrm{Hom}(B_n \hookrightarrow Z_n, M)$

$\Rightarrow$ SES

$$0 \to \underbrace{\operatorname{coker} \partial^{n-1}}_{\cong \mathrm{Ext}(H_{n-1}(C),M)} \to H^n(C;M) \to \underbrace{\ker \partial^n}_{\substack{\cong \mathrm{Hom}(\operatorname{coker} B_n \to Z_n, M) \\ \text{(by Lemma 6)} \\ \cong \mathrm{Hom}(H_n(C),M)}} \to 0$$

because: free res $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$

$\rightsquigarrow$ cochain complex $0 \leftarrow \mathrm{Hom}(B_{n-1},M) \xleftarrow{\partial^{n-1}} \mathrm{Hom}(Z_{n-1},M)$

with $H^1 \cong \operatorname{coker} \partial^{n-1}$, and $H^1 \cong \mathrm{Ext}$ by def of Ext. $\square$

Prop 11 Singular cohomology satisfies axioms that are analogue to the Eilenberg-Steenrod axioms for homology (see ②):

Data

$H^n(-;M)$ are contravariant functors $\{\text{Pairs of Spaces}\} \to R\text{-Mod}$

There are natural connecting homom. $\partial: H^n(A;M) \longrightarrow H^{n+1}(X,A;M)$

Axioms

Homotopy (1) $f \simeq g \Rightarrow f^* = g^*$

Excision (2) $\bar{U} \subseteq A^\circ \Rightarrow \mathrm{incl}^*: H^n(X,A;M) \longrightarrow H^n(X \setminus U, A \setminus U; M)$ iso

Dimension (3) $H^n(\{*\};M) \cong M$ for $n=0$, trivial for $n \neq 0$.

Additivity (4) $H^n(\coprod_\alpha X_\alpha; M) \xrightarrow{i} \prod_\alpha H^n(X_\alpha; M)$ is an iso,

with i given by $i_\alpha = (\text{inclusion } X_\alpha \longrightarrow \coprod_\alpha X_\alpha)^*$.

Exactness (5) There are LESs

$$\cdots \to H^n(X,A;M) \xrightarrow{\mathrm{incl}^*} H^n(X;M) \xrightarrow{\mathrm{incl}^*} H^n(A;M) \xrightarrow{\partial} H^{n+1}(X,A;M) \to \cdots$$

Similarly as for Homology with coefficients, all axioms follow more or less directly from homotopy equivalences of singular chain complexes being send to hom. equiv. of singular cochain complexes by the additive $\mathrm{Hom}(-,M)$ functor.

Proof of (4) Alg Top I: $\sum_\alpha (\mathrm{incl}_\alpha)_c : \bigoplus_\alpha C_\bullet(X_\alpha) \longrightarrow C_\bullet(\coprod_\alpha X_\alpha)$

is a homotopy equivalence $\Rightarrow$ so is

$$\begin{tikzcd}
\mathrm{Hom}(C_\bullet(\coprod_\alpha X_\alpha), M) \arrow[rr, "\mathrm{Hom}(\sum_\alpha (\mathrm{incl}_\alpha)_c\,,\, M)"] \arrow[dd, "\substack{\text{iso} \\ \text{(by def)}}"'] & & \mathrm{Hom}(\bigoplus_\alpha C_\bullet(X_\alpha), M) \arrow[d, "\text{iso}"] \\
& & \prod_\alpha \mathrm{Hom}(C_\bullet(X_\alpha), M) \arrow[d, "\text{iso (by def)}"] \\
C^\bullet(\coprod_\alpha X_\alpha; M) \arrow[rr, "\alpha\text{-component is}"', "(\mathrm{incl}_\alpha)_c"'] & & \prod_\alpha C^\bullet(X_\alpha; M)
\end{tikzcd}$$

$\square$

Further good properties of cohomology:

Thm 12 (Mayer-Vietoris) $A, B \subseteq X$, $A^\circ \cup B^\circ = X \Rightarrow$ LES

$$\dots \to H^n(X;M) \longrightarrow H^n(A;M) \oplus H^n(B;M) \longrightarrow H^n(A\cap B;M) \longrightarrow H^{n+1}(X) \to \dots$$

Remark 13 Understanding the connecting homomorphisms in the Mayer-Vietoris-sequence:

<u>Homology</u> $H_n(X) \longrightarrow H_{n-1}(A\cap B)$:

Represent a homology class $[x] \in H_n(X)$ as $[y+z]$, where $y \in C_n(A)$ and $z \in C_n(B)$. (Here, we abuse notation and write y also for the image of y under $C_n(A) \hookrightarrow C_n(X)$, similarly for z.) Now send $[x] \mapsto [dy]$.

(since $0 = dx = d(y+z) \Rightarrow dy = -dz$, so $dy \in C_{n-1}(A\cap B)$, again abusing notation). See Hatcher p. 150

A similar understanding for cohomology is more complicated. The following wasn't discussed in the lecture.

Cohomology $H^n(A \cap B) \longrightarrow H^{n+1}(X)$:

Extend a cohomology class $[\varphi] \in H^n(A \cap B)$, which is a map $C_n(A \cap B) \to \mathbb{Z}$, to a map $\psi : C_n(A) \to \mathbb{Z}$, ie a cochain $\psi \in C^n(A)$.

Correction 30 April

For each $x \in C_{n+1}(X)$, choose $y \in C_{n+1}(A)$, $z \in C_{n+1}(B)$ such that $x-(y+z)$ is a boundary. Then send $[\varphi]$ to the cohomology class in $H^{n+1}(X)$ that sends each x to $\psi(dy)$.

Thm 14 (Good Pairs) $A \subseteq X$ non-empty closed, A a deformation retract of an open neighbourhood of A in X $\Rightarrow$ the projection $(X,A) \to (X/A, \{*\})$ induces an iso

$$\underbrace{H^n(X/A, \{*\})}_{\cong \tilde{H}^n(X/A)} \longrightarrow H^n(X, A)$$

Def For $X \neq \emptyset$, the n-th reduced cohomology group $\tilde{H}^n(X;M)$ is the n-th cohomology group of the augmented cochain complex

$$0 \to M \xrightarrow{\varepsilon} C^0(X;M) \longrightarrow C^1(X;M) \longrightarrow \dots$$

with $\varepsilon(m)(\sigma) = m$ for all $\sigma : \Delta^0 \to X$.

Prop 15 $H^n(X;M) \cong \tilde{H}^n(X;M)$ for $n \geq 1$,

$H^0(X;M) \cong \tilde{H}^0(X;M) \oplus M$

Ex 16 $\tilde{H}^n(S^k) \cong \mathbb{Z}^{\delta(n,k)}$

$k=0$: ✓. Assume now $k \geq 1$.

1st Proof $C^\bullet_{cw}(S^k) \cong \mathrm{Hom}(C_\bullet^{cw}(S^k), \mathbb{Z}) \cong C_\bullet^{cw}(S^k)$

2nd Proof $H_\bullet(S^k)$ free $\xRightarrow{UCT}$ $H^n(S^k) \cong H_n(S^k)$

3rd Proof $A = S^k \setminus \{e_1\}$, $B = S^k \setminus \{-e_1\}$ $\Rightarrow$ A, B contractible

$\Rightarrow$ Mayer-Vietoris gives iso $H^i(\underbrace{A \cap B}_{\simeq S^{k-1}}) \longrightarrow H^{i+1}(S^k)$

Proceed by induction.

4th Proof

$$\begin{tikzcd}
H^i(S^k) \arrow[d, "\text{iso}"', "\text{LES of Pair } (D^{k+1}, S^k)"] \\
H^{i+1}(D^{k+1}, S^k) \arrow[d, "\text{iso}"', "\text{due to good pair}"] \\
H^{i+1}(S^{k+1})
\end{tikzcd}$$

Prop 17 Let $n \geq 1$. If $f: S^n \to S^n$ has degree $k \in \mathbb{Z}$, then $f^*: H^n(S^n) \longrightarrow H^n(S^n)$ is multiplication by k.

Reminder "f has degree k" is by def equivalent to:

$f_*: H_n(S^n) \to H_n(S^n)$ is multiplication by k

1st Proof

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & C_n^{cw}(S^n) \arrow[r, "0"] \arrow[d, "f_c = \text{mult by } k"] & \cdots \\
\cdots \arrow[r, "0"] & C_n^{cw}(S^n) \arrow[r, "0"] & \cdots
\end{tikzcd}
\xrightarrow[\mathrm{Hom}(\cdot, \mathbb{Z})]{\text{apply functor}}
\begin{tikzcd}
\cdots & C^n_{cw}(S^n) \arrow[l, "0"'] \arrow[d, "f^c = \mathrm{Hom}(f_c, \mathbb{Z}) = \text{mult by } k"] & \cdots \arrow[l, "0"'] \\
\cdots & C^n_{cw}(S^n) \arrow[l, "0"'] & \cdots \arrow[l, "0"']
\end{tikzcd}$$

2nd Proof Use naturality of UCT. (Skipped in lecture)

$\mathrm{Ext}(H_{n-1}(S^n), \mathbb{Z}) \cong 0$ since $H_{n-1}(S^n)$ is free (namely, it is 0 (if $n \geq 2$) or $\mathbb{Z}$ (if $n=1$)). So we have an iso

$$ev : H^n(S^n) \longrightarrow \mathrm{Hom}(H_n(S^n), \mathbb{Z})$$

It is natural, so the following commutes:

$$\begin{tikzcd} H^n(S^n) \arrow[r, "ev", "iso"'] \arrow[d, "f^*"] & \mathrm{Hom}(H_n(S^n), \mathbb{Z}) \arrow[d, "\mathrm{Hom}(f_*, \mathbb{Z}) = \text{mult by } k"] \\ H^n(S^n) \arrow[r, "ev", "iso"'] & \mathrm{Hom}(H_n(S^n), \mathbb{Z}) \end{tikzcd}$$

$\square$

6 The cup product

Reminder about simplexes If $v_0, \dots, v_n \in \mathbb{R}^\ell$ s.t. $v_1 - v_0, \dots, v_n - v_0$ are lin indep., then the convex hull of $\{v_0, \dots, v_n\}$, ie

$$\left\{ \sum_{i=0}^{n} \lambda_i v_i \;\middle|\; \sum_{i=0}^{n} \lambda_i = 1 ,\ (\lambda_0, \dots, \lambda_n) \in [0,1]^{n+1} \right\} \subseteq \mathbb{R}^\ell$$

together with the tuple $(v_0, \dots, v_n)$, is called an **n-simplex**, denoted **$[v_0, \dots, v_n]$**. Every pair of n-simplexes $[v_0, \dots, v_n]$, $[v_0', \dots, v_n']$ is naturally homeomorphic via $\sum \lambda_i v_i \longmapsto \sum \lambda_i v_i'$.

The **standard n-simplex** is $\Delta^n := [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$.

A **singular n-simplex** of a top. space X is a cont. map $\sigma : \Delta^n \to X$.

They form the basis of $C_n(X)$. The **boundary operator** $d : C_n(X) \to C_{n-1}(X)$ is given by $d(\sigma) = \sum_{i=0}^{n} \sigma|_{\underbrace{[e_0, \dots, \hat{e}_i, \dots, e_n]}_{\text{means } e_i \text{ is left out}}}$.

(where we implicitly identify the non-standard simplex $[e_0, \dots, \hat{e}_i, \dots, e_n]$ with Δ^{n-1} via the natural homeo).

Throughout, let R be a commutative unital ring.

Def X top space, $\varphi \in C^n(X;R)$, $\psi \in C^k(X;R)$.

Let the **cup-product $\varphi \smile \psi$** $\in C^{n+k}(X;R)$

\smile, not \cup, in LaTeX

be given sending singular simplexes $\sigma : \Delta^{n+k} = [e_0, \dots, e_{n+k}] \to X$ to

$$(\varphi \smile \psi)(\sigma) = \varphi\Big(\underbrace{\sigma|_{[e_0, \dots, e_n]}}_{\text{"front face" of } \sigma}\Big) \underset{\text{multiplication in } R}{\cdot} \psi\Big(\underbrace{\sigma|_{[e_n, \dots, e_{n+k}]}}_{\text{"back face" of } \sigma}\Big)$$

Prop 1 (1) $\smile : C^n(X;R) \times C^k(X;R) \longrightarrow C^{n+k}(X;R)$

is R-bilinear. (uses distributivity & associativity of R)

(2) $\smile$ is associative: $(\varphi \smile \psi) \smile \eta = \varphi \smile (\psi \smile \eta)$

(uses associativity of R)

(3) Let $\varepsilon \in C^0(X;R)$, $\varepsilon(\sigma) = 1 \in R$ for all σ. Then

$\varphi \smile \varepsilon = \varepsilon \smile \varphi = \varphi$. (uses unit of R)

Proof Exercise.

Remark 2 $\smile$ makes $C^\bullet(X;R) = \bigoplus_{n=0}^{\infty} C^n(X;R)$ into a

(generally non-commutative) unital R-algebra (by Prop 1).

Moreover, $C^\bullet(X;R)$ is graded:

a grading on an R-algebra S is a decomposition

$S = \bigoplus_{n \in \mathbb{Z}} S_n$ as an R-module, such that $S_n S_k \subseteq S_{n+k}$.

We write $\deg x = n$ for $x \in S_n$, $x \neq 0$. deg is not defined if $x \notin S_n \; \forall n$.

Example 3 $C^\bullet(\emptyset; R)$ = the zero ring

$C^\bullet(\{*\}; R)$: For all $n \geq 0$, $C_n(\{*\})$ is generated by the

constant $\sigma_n : \Delta^n \to \{*\}$, and $C^n(\{*\};R)$ by $\varphi_n : \sigma_n \longmapsto 1$.

Check $\varphi_n \smile \varphi_k = \varphi_{n+k}$. So we have an isomorphism of graded

R-algebras: $C^\bullet(\{*\};R) \longrightarrow R[x]$, $\varphi_n \longmapsto x^n$.

Here, deg on $R[x]$ is different from the usual deg of polynomials:

$\deg(r x^n) = n$, deg not defined for non-monomials.

Prop 4 (Graded Leibniz rule). For $\varphi \in C^n(X;R)$, $\psi \in C^k(X;R)$:

$$d(\varphi \smile \psi) = (d\varphi) \smile \psi + (-1)^n \varphi \smile d\psi$$

Koszul sign rule:

"when d jumps over something of degree k, $(-1)^k$ appears"

Proof Calculate:

$$((d\varphi)\smile\psi)(\sigma:[e_0,\dots,e_{n+k+1}]\to X)$$

$$= (d\varphi)(\sigma|_{[e_0,\dots,e_{n+1}]}) \cdot \psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

$$= \varphi(d\sigma|_{\dots}) \cdot \psi(\dots)$$

$$= \varphi\Big(\sum_{i=0}^{n+1}(-1)^i\,\sigma|_{[e_0,\dots,\hat{e}_i,\dots e_{n+1}]}\Big) \cdot \psi(\dots)$$

$$= \sum_{i=0}^{n+1}(-1)^i\,\varphi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots e_{n+1}]})\,\psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

and:

$$(\varphi\smile d\psi)(\sigma) =$$

$$= \sum_{j=0}^{k+1}(-1)^j\,\varphi(\sigma|_{[e_0,\dots,e_n]})\,\psi(\sigma|_{[e_n,\dots,\widehat{e_{n+j}},\dots,e_{n+k+1}]})$$

Now plug this into:

$$((d\varphi)\smile\psi)(\sigma) + (-1)^n(\varphi\smile d\psi)(\sigma)$$

Notice the last summand ($i=n+1$) cancels the first ($j=0$)!

$$= \sum_{i=0}^{n}(-1)^i\,\varphi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{n+1}]})\,\psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

$$+ \sum_{m=n+1}^{n+k+1}(-1)^m\,\varphi(\sigma|_{[e_0,\dots,e_n]})\,\psi(\sigma|_{[e_n,\dots,\hat{e}_m,\dots,e_{n+k+1}]})$$

← index shift $m = j+n$

$$= (d(\varphi\smile\psi))(\sigma)$$

□

Prop 5 (1) cocycle $\smile$ cocycle $=$ cocycle

(2) coboundary $\smile$ cocycle $=$ coboundary and

cocycle $\smile$ coboundary $=$ —— " ——

(3) For $[\varphi] \in H^n(X;R)$, $[\psi] \in H^k(X;R)$,

$[\varphi] \smile [\psi] := [\varphi \smile \psi] \in H^{n+k}(X;R)$ is well-def

(4) $\smile$ makes $H^\bullet(X;R) := \bigoplus_{i=0}^{\infty} H^i(X;R)$ into a graded R-algebra.

Proof (1) If $d\varphi = d\psi = 0 \Rightarrow d(\varphi \smile \psi) = (d\varphi) \smile \psi \pm \varphi \smile d\psi = 0$.

(2) If $\varphi = d\eta$ and $d\psi = 0 \Rightarrow \varphi \smile \psi = (d\eta) \smile \psi = d(\eta \smile \psi)$.

(3) $\varphi \smile \psi$ is a cocycle by (1).

If $\varphi' = \varphi + d\eta$, $\psi' = \psi + d\xi$, then

$$[\varphi' \smile \psi'] = [\varphi \smile \psi] + \underbrace{[\varphi \smile d\xi]}_{=0} + \underbrace{[d\eta \smile \psi']}_{=0} \quad \text{by (2)}$$

(4) Follows from Prop 1 $\square$

Example 6 If $\ell \geq 1$, then $H^\bullet(S^\ell; R) \cong R[x]/(x^2)$ with $\deg x = \ell$ ($x^2 = 0$ since since there is no non-trivial cohomology class of deg 2ℓ).

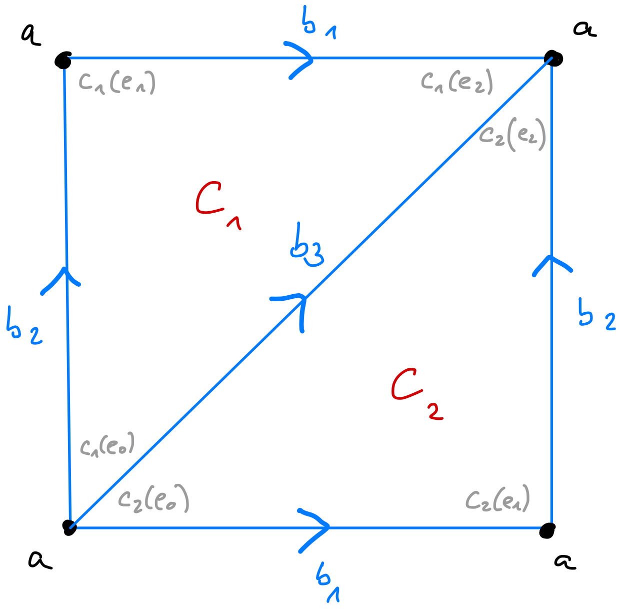
Def For a Δ-complex X, define $\smile$ in the same way as before on the simplicial cochain complex $C^\bullet_\Delta(X; R) = \mathrm{Hom}(C^\Delta_\bullet(X), R)$, and on its cohomology $H^\bullet_\Delta(X; R)$.

Prop 7 The chain homotopy equivalence **Thm 2.27 in Hatcher**
$C^\Delta_\bullet(X) \longrightarrow C_\bullet(X)$, sending simplex to simplex, induces a chain homotopy equivalence $C^\bullet(X) \longrightarrow C^\bullet_\Delta(X)$ that preserves the cup product.

Proof Immediate from def $\square$

Example 8 $X = S^1 \times S^1$. Know $H^0(X) \cong \mathbb{Z}$, $H^1(X) \cong \mathbb{Z}^2$, $H^2(X) \cong \mathbb{Z}$. So $\smile$ may be interesting on $H^1(X)$.

Put a Δ-complex-structure on X:



$a \in C^\Delta_0(X)$, $b_1, b_2, b_3 \in C^\Delta_1(X)$
$c_1, c_2 \in C^\Delta_2(X) \implies$

$db_i = 0$,

$dc_1 = dc_2 = b_1 - b_3 + b_2$

One computes that:

$H^\Delta_0(X; \mathbb{Z})$ has basis $[a]$

$H^\Delta_1(X; \mathbb{Z})$ — " — $[b_1], [b_2]$

$H^\Delta_2(X; \mathbb{Z})$ — " — $[c_1 - c_2]$

Since $H_\bullet^\Delta(X;\mathbb{Z})$ is torsion-free, the UCT implies $H^\bullet_\Delta(X;\mathbb{Z}) \cong \mathrm{Hom}(H_\bullet^\Delta(X;\mathbb{Z}))$. So the dual basis of the basis $[a],[b_1],[b_2],[c_1-c_2]$ is a basis for $H^\bullet_\Delta(X;\mathbb{Z})$:

$$[\varphi],\ [\psi^1],\ [\psi^2],\ [\eta]$$

deg $\quad 0 \quad 1 \quad 1 \quad 2$

with $\varphi(a)=1$, $\psi^i(b_j)=\delta_{ij}$, $\eta(c_1-c_2)=1$.

Let's calculate $[\psi^1]\smile[\psi^2]$! Since $[\psi^1]\smile[\psi^2]\in H^2(X;\mathbb{Z})$

$$\Rightarrow [\psi^1]\smile[\psi^2] = \lambda[\eta] \text{ for some } \lambda\in\mathbb{Z}.$$

Evaluate both sides on $[c_1-c_2]$:

$$\begin{aligned}
\lambda &= ev([\psi^1]\smile[\psi^2])([c_1-c_2]) \\
&= ev([\psi^1\smile\psi^2])([c_1-c_2]) && \text{by def of } \smile \text{ on cohomology} \\
&= (\psi^1\smile\psi^2)(c_1-c_2) && \text{by def of ev} \\
&= (\psi^1\smile\psi^2)(c_1) - (\psi^1\smile\psi^2)(c_2) && \text{by linearity} \\
&= \psi^1(c_1|_{[e_0,e_1]})\,\psi^2(c_1|_{[e_1,e_2]}) - \psi^1(c_2|_{[e_0,e_1]})\,\psi^2(c_2|_{[e_1,e_2]}) \\
& && \text{by def of } \smile \text{ on cochains} \\
&= \psi^1(b_2)\psi^2(b_1) - \psi^1(b_1)\psi^2(b_2) \\
&= -1
\end{aligned}$$

$\Rightarrow [\psi^1]\smile[\psi^2] = -[\eta]$.

Similarly, one computes $[\psi^2]\smile[\psi^1]=[\eta]$

and $[\psi^i]\smile[\psi^i]=0$.

So $H^\bullet(S^1\times S^1;\mathbb{Z}) \cong \underbrace{\mathbb{Z}\langle x,y\rangle}_{\text{free algebra generated by } x,y.} / (xy=-yx,\ x^2=y^2=0)$

Prop 9 (Naturality of $\smile$)

$f: X \to Y$ cont. map of top. spaces, $[\varphi] \in H^n(Y;R)$, $[\psi] \in H^k(Y;R)$

$$\Rightarrow f^*([\varphi] \smile [\psi]) = (f^*[\varphi]) \smile (f^*[\psi])$$

Proof (skipped in the lecture)

For all $(n+k)$-simplexes σ: $(f^c(\varphi \smile \psi))(\sigma) = \varphi \smile \psi(f \circ \sigma)$

$$= \varphi(f \circ \sigma|_{[e_0,\dots,e_n]})\,\psi(f \circ \sigma|_{[e_n,\dots,e_{n+k}]})$$

$$= f^c\varphi(\sigma|_{\dots}) \cdot f^c\psi(\sigma|_{\dots}) = ((f^c\varphi) \smile (f^c\psi))(\sigma).$$

Now $f^*([\varphi] \smile [\psi]) = f^*([\varphi \smile \psi]) = [f^c(\varphi \smile \psi)]$

$= [(f^c\varphi) \smile (f^c\psi)] = [f^c\varphi] \smile [f^c\psi] =$

$f^*([\varphi]) \smile f^*([\psi])$ $\square$

In other words: f^* is a homomorphism of graded R-algebras!

Prop 10 X, Y top spaces $\Rightarrow$ We have graded R-algebra isos

(1) $H^\bullet(X \sqcup Y; R) \xrightarrow{\binom{incl^*}{incl^*}} H^\bullet(X;R) \times H^\bullet(Y;R)$

(2) $H^\bullet(X \vee Y; R) \xrightarrow{\binom{incl^*}{incl^*}}$ Subalgebra of $H^\bullet(X;R) \times H^\bullet(Y;R)$ containing in deg 0 only those (φ, ψ) with $\varphi(x_0) = \psi(y_0)$

wedge product $X \sqcup Y/_{\{x_0\} \sim \{y_0\}}$ for some $x_0 \in X$, $y_0 \in Y$ that are deformation retracts of neighbourhoods N_X, N_Y.

Proof (1) We know $\binom{incl^*}{incl^*}$ is an R-module isom. (eg use MV).

It's an algebra homom by Prop 9.

(2) Mayer-Vietoris gives isos for $n \geq 1$, and a SES

$$0 \to H^0(X \vee Y; R) \to H^0(\underbrace{X \cup N_Y}_{\simeq X}; R) \oplus H^0(\underbrace{Y \cup N_X}_{\simeq Y}; R) \longrightarrow H^0(\underbrace{N_X \cap N_Y}_{\simeq \{*\}}; R) \to 0$$

the kernel is the desired Subalgebra $\square$

Example 11 $H^\bullet(S^1 \vee S^1 \vee S^2) \cong$

$\mathbb{Z}\langle x_1, x_2, x_3\rangle / (x_i x_j = 0 \text{ for all } i,j)$

$\deg x_1 = \deg x_2 = 1, \ \deg x_3 = 2$

This is not isomorphic to the ring $H^\bullet(S^1 \times S^1)$, which contains elements of degree 1 with non-zero product.

$\Rightarrow S^1 \vee S^1 \vee S^2 \not\simeq S^1 \times S^1$

Theorem 13 X top. space, $A \subseteq X$, $\varphi \in H^n(X, A; R)$, $\psi \in H^k(X, A; R)$. Then

$$\varphi \smile \psi = (-1)^{nk} \psi \smile \varphi$$

Proof: next lecture.

This property of the graded R-alg. $H^\bullet(X, A; R)$ is called **graded commutative**.

Theorem 13 X top. space, $[\varphi] \in H^n(X;R)$, $[\psi] \in H^k(X;R)$. Then

Hatcher Thm 3.11, p. 210

$$[\varphi] \smile [\psi] = (-1)^{nk} [\psi] \smile [\varphi].$$

Proof For $\sigma : \Delta^n \to X$, let $\bar{\sigma} : \Delta^n \to X$ be $\bar{\sigma} = \sigma \circ (\text{natural homeo } [e_0, \dots, e_n] \to [e_n, e_{n-1}, \dots, e_1, e_0])$, i.e. $\bar{\sigma}(e_i) = \sigma(e_{n-i})$. Let $\rho : C_\bullet(X) \to C_\bullet(X)$, $\sigma \mapsto (-1)^{\varepsilon_n} \bar{\sigma}$, where $\varepsilon_n = \frac{(n+1)n}{2}$.

<u>Claim 1</u>: ρ is a chain map.

<u>Claim 2</u>: $\rho \simeq \mathrm{id}_{C_\bullet(X)}$

<u>Pf that Claim 1 & 2 $\Rightarrow$ Thm:</u>

$$(\rho^*(\varphi \smile \psi))(\sigma) = (-1)^{\varepsilon_{n+k}} \varphi(\sigma|_{[e_{n+k}, \dots, e_k]}) \, \psi(\sigma|_{[e_k, \dots, e_0]})$$

$$((\rho^*\psi) \smile (\rho^*\varphi))(\sigma) = (-1)^{\varepsilon_n + \varepsilon_k} \psi(\sigma|_{[e_k, \dots, e_0]}) \, \psi(\varphi|_{[e_{n+k}, \dots, e_k]})$$

$$\Rightarrow [\varphi] \smile [\psi] = [\varphi \smile \psi] = [\rho^*(\varphi \smile \psi)]$$

$$= (-1)^{\varepsilon_{n+k} + \varepsilon_n + \varepsilon_k} [(\rho^*\psi) \smile (\rho^*\varphi)] = (-1)^{nk} [\rho^*\psi] \smile [\rho^*\varphi]$$

$$= (-1)^{nk} [\psi] \smile [\varphi].$$

Check that $\varepsilon_{n+k} + \varepsilon_n + \varepsilon_k \equiv nk \ (2)$ ✓

<u>Pf of Claim 1:</u>

$$\rho\, d\sigma = \rho\left(\sum_{i=0}^{n} (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}\right)$$

$$= \sum_{i=0}^{n} (-1)^{i + \varepsilon_{n-1}} \sigma|_{[e_n, \dots, \hat{e}_i, \dots, e_0]}$$

$$d\rho\sigma = \sum_{j=0}^{n} (-1)^{j + \varepsilon_n} \sigma|_{[e_n, \dots, \widehat{e_{n-j}}, \dots, e_0]} \qquad n-j = i$$

$$= \sum_{i=0}^{n} (-1)^{n-i+\varepsilon_n} \sigma|_{[e_n, \dots, \hat{e}_i, \dots e_0]}$$

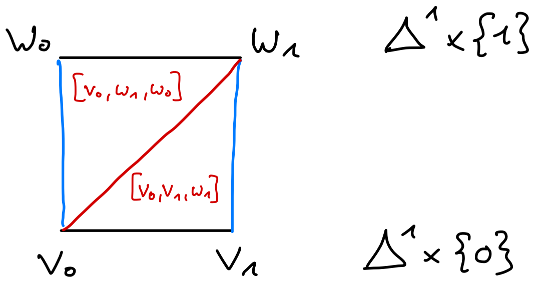
Check: $\varepsilon_{n-1} \equiv n + \varepsilon_n \ (2) \iff n + \frac{n(n-1)}{2} \equiv \frac{n(n+1)}{2}$ ✓

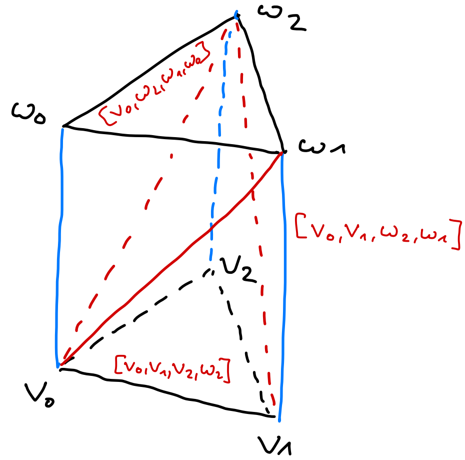
<u>Pf of Claim 2:</u> Need homotopy $s: C_n(X) \longrightarrow C_{n+1}(X)$ with

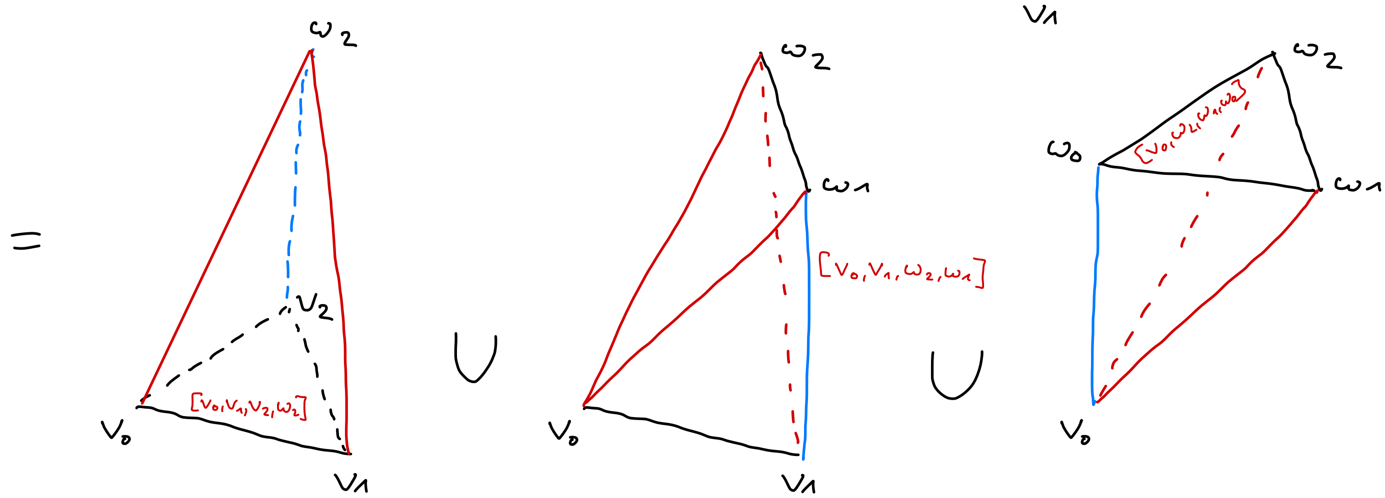
$$d_{n+1} s_n + s_{n-1} d_n = \rho_n - \mathrm{id}_{C_n}. \qquad (*)$$

Construction of s is inspired by the prism operator:
cut the prism $\Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$
into $n+1$ many $(n+1)$-simplices.

Let $v_i = (e_i, 0)$ and $w_i = (e_i, 1)$ for $i = 0, \dots, n$.







Let $\pi: \Delta^n \times [0,1]$ be the projection, so that $\pi(w_i) = \pi(v_i) = e_i$.

Define

$$\boxed{s_n(\sigma) := \sum_{i=0}^{n} (-1)^{i+\varepsilon_{n-i}} \sigma \circ \pi([v_0, \dots, v_i, w_n, \dots, w_i])}$$

Let us check by calculation that $(*)$ holds.

$$d_{n+1}(S_n(\sigma)) \overset{(1)}{=} \sum_{0\le j\le i\le n} (-1)^{i+\varepsilon_{n-i}+j}\ \sigma\circ\pi([v_0,\dots,\hat{v}_j,\dots,v_i,w_n,\dots,w_i])$$

$$+ \overset{(2)}{\sum_{0\le i\le j\le n}} (-1)^{\varepsilon_{n-i}+n+j+1}\ \sigma\circ\pi([v_0,\dots,v_i,w_n,\dots,\underset{\substack{\uparrow\\ \text{index } n-j+i+1}}{\hat{w}_j},\dots w_i])$$

Consider the summands with $i=j$:

$$(-1)^{\varepsilon_n}\ \sigma\circ\pi([w_n,\dots,w_0]) +$$

$$+ \sum_{i=1}^{n+1} (-1)^{\varepsilon_{n-i}}\ \sigma\circ\pi([v_0,\dots,v_{i-1},w_n,\dots,w_i])$$

$$+ \sum_{k=0}^{n} (-1)^{\varepsilon_{n-k}+n+k+1}\ \sigma\circ\pi([v_0,\dots,v_k,w_n,\dots,w_{k+1}])$$

$$+ (-1)^{\varepsilon_0}\ \sigma\circ\pi([v_0,\dots,v_n])$$

these cancel: index shift $k=i-1$, check $\varepsilon_{n-i} \not\equiv \varepsilon_{n-i+1}+n+i \quad (2)$

$$= (-1)^{\varepsilon_n}\bar{\sigma} + \sigma = \rho\sigma - \sigma$$

So, to prove $(*)$, one has to check that the summands with $i\neq j$ equal $-S_{n-1}(d_n(\sigma))$

$$= -S_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\,\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}\Big)$$

$$= \sum_{0\le j\le k\le n} (-1)^{1+j+k+\varepsilon_{n-k-1}}\ \sigma\circ\pi([v_0,\dots,\hat{v}_j,\dots,v_{k+1},w_n,\dots,w_{k+1}])$$

index shift: $k=i-1$. check $i+\varepsilon_{n-i}+j \equiv 1+j+i-1+\varepsilon_{n-i}$

$\Rightarrow$ equals summands of (1) with $j<i$

$$+ \sum_{0\le i<j\le n} (-1)^{1+j+i+\varepsilon_{n-i-1}}\ \sigma\circ\pi([v_0,\dots,v_i,w_n,\dots\hat{w}_j,\dots w_i])$$

check: $\varepsilon_{n-i}+n+j+1 \equiv 1+j+i+\varepsilon_{n-i-1}$

$\Rightarrow$ equals summands of (2) with $i<j$

$\square$

Remark 14 We'll prove later that:

$$H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1}) \qquad \text{with } \deg x = 2$$

(commutative since $H^k(\mathbb{C}P^n) = 0$ for odd k)

$$H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1}) \quad \text{with } \deg x = 1$$

(commutative because of $\mathbb{Z}/2$ coefficients)

$$H^\bullet((S^1)^{\times n}) \cong \mathbb{Z}\langle x_1, \dots, x_n\rangle / (x_i x_j + x_j x_i,\ x_i^2)$$

with $\deg x_i = 1$

(not commutative, but graded commutative)

Reminder from Alg Top 1 X top. space, $A, B \subseteq X$.

$C_n(A+B) \subseteq C_n(A \cup B)$ is generated by $C_n(A) \cup C_n(B) \subseteq C_n(A \cup B)$.

$C_n(A+B)$ is a chain complex, and $C_\bullet(A+B) \xhookrightarrow{i} C_\bullet(A \cup B)$

is a homotopy equivalence (proved by barycentric subdivision).

Lemma 14 There is a (natural) iso

$H^n(X, A \cup B; R) \xrightarrow{j} H^n(X, A+B; R)$ induced by i.

Proof (skipped in lecture)

$$\begin{array}{ccccccccc}
0 \to & C_n(A+B) & \to & C_n(X) & \to & C_n(X, A+B) & \to 0 \\
& \downarrow i & & \downarrow \mathrm{id}_X & & \downarrow & \\
0 \to & C_n(A \cup B) & \to & C_n(X) & \to & C_n(X, A \cup B) & \to 0
\end{array}$$

Commutes, has split exact rows. Apply $\mathrm{Hom}(-, R)$ and take the natural LESs in cohomology:

$$\begin{array}{ccccccccc}
\cdots \leftarrow & H^n(A+B; R) & \leftarrow & H^n(X; R) & \leftarrow & H^n(X, A+B; R) & \leftarrow \cdots \\
& \text{iso} \uparrow i^* & & \uparrow \mathrm{id} & & \uparrow j & \\
\cdots \leftarrow & H^n(A \cup B; R) & \leftarrow & H^n(X; R) & \leftarrow & H^n(X, A \cup B); R & \leftarrow \cdots
\end{array}$$

j is an iso by the five lemma. $\square$

Def Let X be a top. space and $A, B \subseteq X$. Let the **relative cup product**

$$\smile : H^n(X, A; R) \times H^k(X, B; R) \longrightarrow H^{n+k}(X, A \cup B; R)$$

be the postcomposition with j^{-1} of the bilinear map on cohomology induced by

$$\smile : C^n(X, A; R) \times C^k(X, B; R) \longrightarrow C^{n+k}(X, A+B; R)$$

$$(\varphi \smile \psi)(\underset{\substack{\uparrow \\ \text{im } \sigma \subseteq A \text{ or} \\ \text{im } \sigma \subseteq B}}{\sigma}) = \underbrace{\varphi(\sigma|_{[e_0, \dots, e_n]})}_{0 \text{ if im } \sigma \subseteq A} \underbrace{\psi(\sigma|_{[e_n, \dots, e_{n+k}]})}_{0 \text{ if im } \sigma \subseteq B}$$

(I) Motivation

Def (Poincaré algebra) A connected ($\Leftrightarrow A^0 = \Bbbk$) gca $A^\bullet = \bigoplus_{j=0}^{\infty} A^j$ over a field $\Bbbk$ is called a **Poincaré algebra** of formal dimension n if.

(i) $A^j = 0$ for $j > n$.

(ii) $A^n \cong \Bbbk$

(iii) the bilinear pairing $A^j \otimes A^{n-j} \longrightarrow A^n \cong \Bbbk$ is non-degenerate
$\Longleftrightarrow$ the map $A^j \longrightarrow \mathrm{Hom}_{\Bbbk}(A^{n-j}, \Bbbk)$ is an isomorphism.

Claim Let M^n be a closed connected orientable manifold.
Then $H^\bullet(M;\mathbb{Q})$ is a Poincaré algebra of formal dimension n.

(II) Manifolds

Def (Topological manifold) A Hausdorff second countable topological space M is called a **topological manifold** (resp. **top. mnfd with boundary**) of dimension n if each point $x \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (resp. of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$).

Def (Boundary) Let M be a manifold with boundary. The subset ∂M of points $x \in M$ that do not have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ is called the boundary of M.

Def (Closed manifold) A compact manifold without boundary is called **closed**.

Examples: (i) $\mathbb{R}^n$ any any open subset of $\mathbb{R}^n$.

(ii) $S^n := \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} (x^i)^2 = 1\}$

Two charts: $\varphi_n : S^n \setminus \{n\} \longrightarrow \mathbb{R}^n$

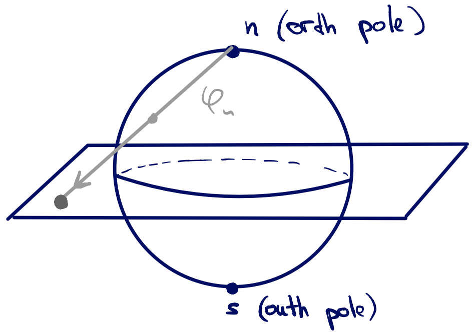
$$(x^0, \dots, x^n) \longmapsto \left(\frac{x^0}{1-x^n}; \dots; \frac{x^{n-1}}{1-x^n}\right)$$

$\varphi_s : S^n \setminus \{s\} \longrightarrow \mathbb{R}^n$

$$(x^0, \dots, x^n) \longmapsto \left(\frac{x^0}{1+x^n}; \dots; \frac{x^{n-1}}{1+x^n}\right)$$

with transition maps: $\varphi_s \circ \varphi_n^{-1}, \varphi_n \circ \varphi_s^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$.

$$(t^1, \dots, t^n) \longmapsto \left(\frac{t^1}{\|t\|^2}; \dots; \frac{t^n}{\|t\|^2}\right)$$



(iii) n-dimensional torus T^n;

(iv) real and complex projective spaces $\mathbb{RP}^n$ & $\mathbb{CP}^n$.

... **with boundary:** (i) D^n;

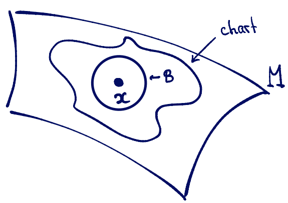
(ii) solid torus $S^1 \times D^2$.

Non-examples: (i) 

(ii) $\mathbb{RP}^\infty = \bigcup_{n=0}^{\infty} \mathbb{RP}^n$ & $\mathbb{CP}^\infty = \bigcup_{n=0}^{\infty} \mathbb{CP}^n$.

Proposition 1 Let M^n be a topological manifold. Then for any $x \in M$: $H_i(M, M \setminus \{x\}; R) \cong \begin{cases} 0 & \text{if } i \neq n; \\ R & \text{if } i = n. \end{cases}$

$\triangleright$ Let B be an open ball around x (sits inside of a neighborhood of x homeomorphic to a subset of $\mathbb{R}^n$).



$\Longrightarrow Z := M \setminus B$ is closed.

By excision theorem, $H_i(M, M\setminus\{x\}; R) \cong H_i(M\setminus Z; (M\setminus\{x\})\setminus Z; R) \cong H_i(B, B\setminus\{x\}; R)$

$$\Rightarrow \cdots \to \underbrace{\tilde{H}_i(B;R)}_{=0} \to H_i(B, B\setminus\{x\}; R) \xrightarrow{\cong} \underbrace{\tilde{H}_{i-1}(B\setminus\{x\}; R)}_{\cong\, \tilde{H}_{i-1}(S^n; R)} \to \underbrace{\tilde{H}_{i-1}(B;R)}_{=0} \to \cdots$$

$\square$

Def (Local (co)homology) $H_i(M, M\setminus\{x\}; R)$ local homology group
$H^i(M, M\setminus\{x\}; R)$ local cohomology group

Def (Homology manifold) A Hausdorff second countable space is a homology R-manifold of dimension n if for any $x \in M$ $H_\bullet(M, M\setminus\{x\}; R) \cong \tilde{H}_\bullet(S^n; R)$.

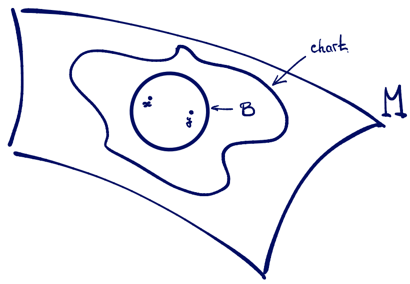
III Orientations

Def (Local orientations) A local orientation μ_x in $x \in M$ is a generator of the local homology group $H_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$.

Note that there are two choices of a generator in $\mathbb{Z}$.
$\Rightarrow$ At each point there are two possible orientations.

Def (Orientation) An orientation of an n-dimensional manifold is a choice of a local orientation $\mu_x \in H_n(M, M\setminus\{x\}; \mathbb{Z})$ at every $x \in M$, s.t. it is locally consistent, i.e. if $x, y \in M$ can be covered by a ball B within one chart, then μ_x and μ_y map one to each other under the isomorphisms:

$$H_n(M, M\setminus\{x\}; \mathbb{Z}) \xleftarrow{\cong} H_n(M, M\setminus B; \mathbb{Z}) \xrightarrow{\cong} H_n(M, M\setminus\{y\}; \mathbb{Z})$$



Def ((non-) Orientable manifold) A manifold is orientable if there exists an orientation on M.
A manifold is non-orientable if it is not orientable.

Examples: (i) S^n is orientable.
(ii) The Möbius band is non-orientable.

Proposition 2 Let M be a closed connected manifold of dimension n.
(i) The homomorphism $H_n(M; \mathbb{F}_2) \to H_n(M, M\setminus\{x\}; \mathbb{F}_2)$ is an isomorphism for any $x \in M$.
(ii) If M is orientable, then $H_n(M; \mathbb{Z}) \to H_n(M, M\setminus\{x\}; \mathbb{Z})$ is an isomorphism for any $x \in M$.
If M is non-orientable, then $H_n(M; \mathbb{Z}) = 0$.
(iii) $H_i(M; \mathbb{Z}) = 0$ for $i > n$.

Main Lemma 3 Let $A \subseteq M$ be a compact subset of a manifold M of dimension n. (not necessary compact).
(i) $H_i(M, M\setminus A; R) = 0$ if $i > n$.
$\alpha \in H_n(M, M\setminus A; R)$ is zero iff its image in $H_n(M, M\setminus\{x\}; R)$ is zero for every $x \in A$.
(ii) For every locally consistent choice of orientations $\mu_x, x \in A$, exists a unique $\mu_A \in H_n(M, M\setminus A; R)$ s.t. its image in $H_n(M, M\setminus\{x\}; R)$ is μ_x for all $x \in A$.

▷ **Step 1.** If the assertion holds for compact A, B and $A\cap B$, then it holds for $A\cup B$.

Relative Mayer-Vietoris sequence:

$$\underset{\overset{\parallel}{0}}{H_{n+1}(M, M\setminus(A\cap B))} \longrightarrow H_n(M, M\setminus(A\cup B)) \xrightarrow{\Phi} H_n(M, M\setminus A)\oplus H_n(M, M\setminus B) \xrightarrow{\Psi} H_n(M, M\setminus(A\cap B))$$

For $i>n$ we have $H_i(M, M\setminus(A\cap B)) = H_i(M, M\setminus A) = H_i(M, M\setminus B) = 0$ $\Longrightarrow H_i(M, M\setminus(A\cup B))$ is locked between two zeros $\Rightarrow$ zero itself.

If $\mu\in H_n(M, M\setminus(A\cup B))$ is s.t. $\mu_x\in H_n(M, M\setminus\{x\})$ is zero for all $x\in A\cup B \Longrightarrow$ its images in $H_n(M, M\setminus A)$ and $H_n(M, M\setminus B)$ are zero by the assumption. $\Longrightarrow$ Since Φ is injective, $\mu = 0$. (Proves (i)).

Let μ_x, $x\in A\cup B$ be a locally consistent choice of orientations $\Longrightarrow \exists!$ $\mu_A\in H_n(M, M\setminus A)$, $\mu_B\in H_n(M, M\setminus B)$
$\Psi(\mu_A, \mu_B) = \mu_A|_{A\cap B} - \mu_B|_{A\cap B} \in H_n(M, M\setminus(A\cap B))$. its image is zero in $H_n(M, M\setminus\{x\})$ for any $x\in A\cap B$

$\Longrightarrow$ it is zero itself by assumption on $A\cap B$. $\Longrightarrow$ By exactness, (μ_A, μ_B) is the image of a unique element $\mu_{A\cup B}\in H_n(M, M\setminus(A\cup B))$. since Φ is injective $\boxed{1}$

Step 2. It is enough to prove the assertion for a compact subset of a single chart. (i.e. in $\mathbb{R}^n$)

Any compact subset $A\subseteq M$ is a union of a finite number of compact subsets, s.t. each belongs to a chart $\Longrightarrow$ We can apply induction and **Step 1**.
If U is a chart, then $H_i(M, M\setminus A)\cong H_i(U, U\setminus A)$ by excision. $\boxed{2}$

$\Longrightarrow$ From now on we assume $M = \mathbb{R}^n$.

Step 3 If $A\subseteq\mathbb{R}^n$ is a finite simplicial complex, s.t. its simplices are linearly embedded, then the assertion follows by induction, and it is enough to prove for one simplex. The latter follows from the definition of local consistency. $\boxed{3}$

Step 4. $A\subseteq\mathbb{R}^n$ compact
$\alpha\in H_i(\mathbb{R}^n, \mathbb{R}^n\setminus A)$ is represented by a relative cycle z and let $C\subseteq\mathbb{R}^n\setminus A$ be a union of the images of the singular simplices of ∂z.
A and C are compact $\Longrightarrow$ they have positive distance $\delta>0$ between them.

Cover A with a finite piecewise linear simplicial complex K with $K\cap C=\varnothing$: (i) cover A by one big enough simplex; (ii) take barycentric subdivision s.t. the diameter of a piece is less than δ. (iii) take simplices that intersect A.
from Step 3

The same chain z represents a class $\alpha_K\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus K)$ that maps to $\alpha\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus A)$.
By **Step 3**, $\alpha_K = 0$ for $i>n$ $\Longrightarrow$ $\alpha = 0$ and $H_i(\mathbb{R}^n, \mathbb{R}^n\setminus A) = 0$ for $i>n$.

Finally, assume $i=n$. If $\alpha_{K,x} = 0\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\})$ for all $x\in A$, then it also holds for all $x\in K$.
Indeed, for any simplex $\Delta\in K$ and any $x\in\Delta$ the map $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\Delta)\longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\})$ is an iso.
Step 3 now implies that $\alpha_K = 0 \Longrightarrow \alpha = 0$, which concludes the proof of (i) and uniqueness part in (ii).

Existence: let $\alpha_A\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus A)$ be the image of $\alpha_B\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B)$, where B is a big ball containing A. $\square$
exists by definition of local consistency.

Last time: Proof of 26 April

Lemma 3 M^n without boundary, $A \subseteq M$ compact, R commutative unital ring.

(i) $H_i(M, M \setminus A; R) = 0$ for $i > n$.

$\alpha \in H_n(M, M\setminus A; R)$ is zero $\iff$

image of α in $H_n(M, M \setminus \{x\}; R)$ is zero for all $x \in A$.

(ii) μ_x locally consistent choice of orientation for $x \in A$

$\Rightarrow$ exists unique $\mu_A \in H_n(M, M\setminus A; R)$ mapping to μ_x for all $x \in A$

Today:

Prop 2 M^n closed ($\iff$ compact, no boundary) connected.

(i) $H_n(M; \mathbb{F}_2) \to H_n(M, M\setminus\{x\}; \mathbb{F}_2)$ iso for all $x \in M$.

(ii) M orientable $\Rightarrow$ $H_n(M; \mathbb{Z}) \to H_n(M, M\setminus \{x\}; \mathbb{Z})$ iso $\forall x \in M$.

M non-orientable $\Rightarrow$ $H_n(M; \mathbb{Z}) = 0$.

(iii) $H_i(M; \mathbb{Z}) = 0$ for $i > n$.

Note that (iii) follows from Lemma 3 (i) with $A = M$. For (i) & (ii), we'll also use Lemma 3, but need some more tools.

For M^n without boundary, let Hatcher p. 235

$\tilde{M} := \{ \mu_x \mid x \in M$ and $\mu_x \in H_n(M, M\setminus\{x\})$ a local orientation$\}$

Note $p: \tilde{M} \to M$, $\mu_x \mapsto x$ is a 2:1 surjection. For $B \subseteq$ chart $\subseteq M$ an open ball and a generator $\mu_B \in H_n(M, M \setminus B)$, let

$U(\mu_B) := \{ \mu_x \in \tilde{M} \mid x \in B, \ \mu_x$ image of μ_B under
$H_n(M, M\setminus B) \to H_n(M, M\setminus\{x\}) \}$

Exercise The $U(\mu_B)$ form the base of a topology on $\tilde{M}$, st p is a 2:1 covering.

Def $p: \tilde{M} \to M$ is called the orientation covering of M.

Each $\mu_x \in \tilde{M}$ has a canonical orientation $\tilde{\mu}_x \in H_n(\tilde{M}, \tilde{M} \setminus \mu_x)$ corresponding to μ_x under the isos

$$H_n(\tilde{M}, \tilde{M} \setminus \mu_x) \xleftarrow[\text{excision}]{} H_n(U_{(\mu_B)}, U_{(\mu_B)} \setminus \mu_x)$$
$$\longrightarrow H_n(B, B \setminus x) \xrightarrow[\text{excision}]{} H_n(M, M \setminus x)$$

These are locally consistent, so $\tilde{M}$ has a canonical orientation.

Prop 4 If M is connected, then: $\tilde{M}$ non-connected $\Longleftrightarrow$ M orientable

Proof M has orientation μ_x $\Rightarrow$ $\tilde{M} = \underbrace{\{\mu_x \mid x \in M\}}_{\text{open}} \sqcup \underbrace{\{-\mu_x \mid x \in M\}}_{\text{open}}$

If $\tilde{M}$ has two components N_1, N_2, then they inherit an orientation from $\tilde{M}$. Check that $p|_{N_i} : N_i \longrightarrow M$ are coverings. Then, they must be one-sheeted coverings, i.e. homeomorphisms. $\square$

Example $\widetilde{S^2} \cong S^2 \sqcup S^2$, $\widetilde{\mathbb{R}P^2} \cong S^2$, $\widetilde{\text{Klein Bottle}} \cong S^1 \times S^1$

Note that $S^3 \to \mathbb{R}P^3$ is an orientable double covering, but not the orientation covering, which is $\mathbb{R}P^3 \sqcup \mathbb{R}P^3 \to \mathbb{R}P^3$ (since $\mathbb{R}P^3$ is orientable).

Def A **section** of p is a cont. map $s : M \longrightarrow M_R$ with $ps = \mathrm{id}_M$.

Note that a section of a covering map has a component of M as image

Prop 5 μ_x is an orientation $\Longleftrightarrow$ $x \longmapsto \mu_x$ is a section of p

Pf Exercise $\square$

Def R commutative unital ring, M^n without boundary.

Local R-orientation: μ_x is a generator of $H_n(M, M \setminus x; R)$

R-orientation: locally consistent choice of local R-orientations.

M R-orientable $:\Longleftrightarrow$ There exits an R-orientation

Example Every M is $\mathbb{F}_2$-orientable, since there is precisely one local $\mathbb{F}_2$-orientation at every point.

Def Let $M_R := \{\alpha_x \mid x \in M,\ \alpha_x \in H_n(M, M\setminus\{x\}; R)\}$, with similar topology as $\tilde{M}$.

Note $p_R : M_R \longrightarrow M$ is an $|R|$-sheeted covering.

Prop 6 Let $M_r = \{\alpha_x \mid \alpha_x$ is the image of $\mu_x \otimes r$ under the iso $H_n(M, M\setminus x) \otimes R \longrightarrow H_n(M, M\setminus x; R)$ for μ_x a generator of $H_n(M, M\setminus x)\}$

Then: $M_r \subseteq M_R$ is open ; $M_r = M_{-r}$;

$M_r \cap M_s = \emptyset$ for $r \neq \pm s$;

$M_r \cong M$ if $r = -r$, and $M_r \cong \tilde{M}$ if $r \neq -r$.

Pf: Exercise $\square$

Prop 7 μ_x is an R-orientation $\Longleftrightarrow$ $x \longmapsto \mu_x$ is a section of p_R with each μ_x a generator of $H_n(M, M\setminus x; R)$

Pf Exercise, similar to Prop 5. $\square$

Prop 8 If $0 = 2$ in R $\Rightarrow$ all M^n are R-orientable

If $0 \neq 2$ in R $\Rightarrow$ M^n is R-orientable iff it is $\mathbb{Z}$-orientable

Proof $0 = 2 \Rightarrow M_1 \cong M \Rightarrow p_R$ has a section to $M_1 \Rightarrow M$ is R-orientable

Assume $0 \neq 2$. Generators of $H_n(M, M\setminus x; R)$ are of the form $\mu_x \otimes u$ for μ_x a gen. of $H_n(M, M\setminus x)$ and $u \in R$ a unit. Then $u \neq -u$ $\Rightarrow M_u \cong \tilde{M} \Rightarrow p_R$ has a section to M_u iff $\tilde{M} \to M$ has a section. $\square$

Proof of Prop 2 (i) and (ii) Pointwise sum and pointwise R-multiplication turn $\Gamma(M, M_R)$ into an R-module.

$$H_n(M;R) \longrightarrow \Gamma(M, M_R),$$
$$\alpha \longmapsto (x \longmapsto \text{image of } \alpha \text{ in } H_n(M, M\setminus x; R))$$

is a homomorphism. By Lemma 3, applied to $A = M$, it is an isomorphism! Indeed, Lemma 3 (i) yields injectivity. And Lemma 3 (ii) yields surjectivity (here, we need a slightly more general version of Lemma 3 (ii): namely, for every locally consistent choice $\alpha_x \in H_n(M, M\setminus x; R)$, $\exists!\ \mu_A \in H_n(M, M\setminus A; R)$ that maps to α_x for all x. The proof is the same — we never use that α_x generates).

$$M \ R\text{-orientable} \Rightarrow \left\{ \begin{array}{ll} \tilde{M} = M \sqcup M & \text{if } 0 \neq 2 \\ M_r = M \text{ for all } r \in R & \text{if } 0 = 2 \end{array} \right\} \Rightarrow M_R \cong \coprod_{r \in R} M$$

$\Rightarrow \Gamma(M, M_R) \cong R$ (using connectedness of M) $\Rightarrow H_n(M; R) \cong R$.
So $H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$ for all M (since all M are $\mathbb{F}_2$-orientable), and $H_n(M) \cong \mathbb{Z}$ for all orientable M.

M non-orientable $\Rightarrow \tilde{M}$ is connected $\Rightarrow$

$$M_{\mathbb{Z}} \cong \underbrace{M_0}_{\cong M} \sqcup \underbrace{M_1}_{\cong \tilde{M}} \sqcup \underbrace{M_2}_{\cong \tilde{M}} \cdots$$

So the only section of $p_{\mathbb{Z}}$ goes to $M_0 \Rightarrow \Gamma(M, M_{\mathbb{Z}}) \cong 0$
$\Rightarrow H_n(M) \cong 0$. $\square$