Algebraic Topology II (FS '24, ETHZ) 14 Feb

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Alg Top I Top. Space X

$$\rightsquigarrow$$

Singular Chain Complex $C(X) = \cdots \longrightarrow C_1(X) \xrightarrow{d_1} C_0(X) \longrightarrow 0$

$$\rightsquigarrow$$

Homology groups $H_i(X)$

Alg Top II Spice up $C(X)$ before taking homology to get more sensitive invariants and more geom. applications

Topics:
* Homology with Coefficients (for abelian groups M define chain complex $C(X) \otimes M$ with homology groups $H_i(X;M)$)
* Cohomology (cochain complex $\mathrm{Hom}(C(X), M)$ with cohomology groups $H^i(X;M)$)
* Poincaré Duality for compact n-dim manifolds X ($H_i(X;M) \cong H^{n-i}(X;M)$, leading to intersection forms $H_{n/2}(X) \times H_{n/2}(X) \longrightarrow \mathbb{Z}$ for even n)

Color Scheme: Sections, Date

Def / Thm / Proof etc.

Newly defined terms

References

Corrections

① Tensor Products of modules (Spanier: Intro, Sec 4 & Ch 5 Sec 1; Hatcher Sec 3.2 / A Künneth formula; Atiyah–MacDonald Ch 2 / Tensor Product of modules)

Let R be a commutative ring with 1 (after this section only $R = \mathbb{Z}$).

Prop 1 Let M, N be R-modules. Then there exists an R-module T and a bilinear map $\mu : M \times N \longrightarrow T$ such that:

For all R-modules K and bilinear maps $f : M \times N \longrightarrow K$ there is a unique homomorphism $g : T \to K$ with $g \circ \mu = f$.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & K \\ T \arrow[ur, dashed, "\exists!\, g"'] & \end{tikzcd}$$

Proof $U :=$ free R-module with basis the set $M \times N$.

$I :=$ submodule of U generated by

$$\{(\lambda x + x', y) - \lambda(x,y) - (x', y) \mid \lambda \in R,\ x, x' \in M,\ y \in N\}$$
$$\cup \{(x, \lambda y + y') - \lambda(x,y) - (x, y') \mid \lambda \in R,\ x \in M,\ y, y' \in N\}$$

Let $T = U/I$ and $\mu : M \times N \longrightarrow T$, $\mu(x,y) = [(x,y)]$

Check that μ is bilinear! Now let $f : M \times N \to K$ as above be given.

Check <u>existence of g</u>:

Let $\tilde{g} : U \to K$ be the homomorphism with $\tilde{g}((x,y)) = f(x,y)$.

Check that $I \subseteq \ker \tilde{g} \Rightarrow \tilde{g}$ induces $g : T \longrightarrow K$.

We have $g(\mu(x,y)) = g([(x,y)]) = \tilde{g}((x,y)) = f(x,y)$ ✓

Check <u>uniqueness of g</u>:

If $g' : T \to K$ with $g' \circ \mu = f$, then $g'([(x,y)]) = g([(x,y)])$ for all $x \in M$, $y \in N$. But such $[(x,y)]$ generate T $\Rightarrow g = g'$ ✓

$\square$

Prop 2 If $\mu : M \times N \to T$ and $\mu' : M \times N \to T'$ both satisfy the condition in Prop 1, then there is a unique isomorphism $\varphi : T \to T'$ such that $\varphi \circ \mu = \mu'$.

$$\begin{tikzcd} & M \times N \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] \arrow[d, phantom, "\circlearrowleft"] & \\ T \arrow[rr, "\varphi"'] & & T' \end{tikzcd}$$

Proof By assumption (existence of g), $\exists\, \varphi : T \to T'$ with $\varphi \circ \mu = \mu'$ and $\exists\, \psi : T' \to T$ with $\psi \circ \mu' = \mu$. Then $\psi \circ \varphi : T \to T$ with $\psi \circ \varphi \circ \mu = \mu$. By assumption (uniqueness of g) $\Rightarrow \psi \circ \varphi = \mathrm{id}_T$. Similarly $\varphi \circ \psi = \mathrm{id}_{T'}$. $\square$

Def T as in Prop 1 is called the tensor product of M and N over R, written $M \otimes_R N$. Drop R if there is no ambiguity. Write $x \otimes y = \mu(x, y) \in M \otimes_R N$.

Notation $\times$ and $\oplus$ is the same for finitely many modules.

Prop 3 (1) $\exists$ iso $M \otimes N \to N \otimes M$ with $x \otimes y \mapsto y \otimes x$.

(2) $\exists$ iso $(M \oplus N) \otimes K \to (M \otimes K) \oplus (N \otimes K)$ with $(x, y) \otimes z \mapsto (x \otimes z) + (y \otimes z)$

(3) $I \subseteq R$ ideal $\Rightarrow \exists$ iso $(R/I) \otimes M \to M/IM$ with $r \otimes m \mapsto [rm]$

Rmk 4 Special case of (3): Iso $R \otimes M \to M$, $r \otimes m \mapsto rm$.

Proof of Prop 3 (1)

$$\begin{tikzcd} M\oplus N \arrow[r, "h:(x,y)\mapsto(y,x)"] \arrow[d, "\mu_{M,N}"'] \arrow[dr, phantom, "\circlearrowleft"] & N\oplus M \arrow[d, "\mu_{N,M}"] \\ M\otimes N \arrow[r, dashed, "\exists!\varphi"'] & N\otimes M \end{tikzcd}$$

Let $h: M\oplus N \to N\oplus M$ be the homom. with $(x,y)\longmapsto(y,x)$.

Then $\mu_{N,M}\circ h: M\oplus N \longrightarrow N\otimes M$ is bilinear.

By the universal property of $\otimes$, $\exists\ \varphi: M\otimes N \longrightarrow N\otimes M$ with

$\varphi\circ\mu_{M,N} = \mu_{N,M}\circ h$, ie $\varphi(x\otimes y) = y\otimes x$.

Let ψ be the analoguous homo with M, N switched $\Rightarrow$

φ, ψ are mutually inverse homomorphisms.

In the lecture, a similar (but incorrect) proof was given, based on the erroneous assumption that h is <u>bilinear</u> (it is, in fact, <u>linear</u>).

Proof of (2)–(4): Exercises. $\square$

Rmk 5 Using Prop 3, we can calculate $M\otimes_{\mathbb{Z}} N$ for all finitely generated abelian groups M, N.

Example 6 $\mathbb{Z}^2\otimes\mathbb{Z}^2 = (\mathbb{Z}\oplus\mathbb{Z})\otimes(\mathbb{Z}\oplus\mathbb{Z}) \cong \mathbb{Z}^4$

So $\mathbb{Z}^2\otimes\mathbb{Z}^2$ is free with basis $e_1\otimes e_1$, $e_1\otimes e_2$, $e_2\otimes e_1$, $e_2\otimes e_2$.

Careful! Not every element of $\mathbb{Z}^2\otimes\mathbb{Z}^2$ is of the form $x\otimes y$,

eg $e_1\otimes e_1 + e_2\otimes e_2$ isn't (and isn't equal to $(e_1+e_2)\otimes(e_1+e_2)$).

Rmk 7 (1) Every element of $M\otimes N$ is equal to $\sum_{i=1}^{n} x_i\otimes y_i$ for some finite n, $x_i\in M$, $y_i\in N$.

(2) $(\lambda x)\otimes y = x\otimes(\lambda y)$

(3) $(x+x')\otimes y = x\otimes y + x'\otimes y$