

Algebraic Topology II (FS '24, ETHZ)

14 Feb

1

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Alg Top I Top. Space X

\Downarrow

Singular Chain Complex $C(X) = \dots \rightarrow C_n(X) \xrightarrow{d_n} C_{n-1}(X) \rightarrow \dots \rightarrow C_0(X) \rightarrow 0$

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Homology groups $H_i(X)$

Alg Top II

Spice up $C(X)$ before taking homology

to get more sensitive invariants and more geom. applications

Topics: * Homology with Coefficients (for abelian groups M define

chain complex $C(X) \otimes M$

with homology groups $H_i(X; M)$)

* Cohomology (cochain complex $\text{Hom}(C(X), M)$ with

cohomology groups $H^i(X; M)$)

* Poincaré Duality for compact n -dim manifolds X

($H_i(X; M) \cong H^{n-i}(X; M)$, leading to

intersection forms $H_{n/2}(X) \times H_{n/2}(X) \rightarrow \mathbb{Z}$ for even n)

Color Scheme: Sections, Date

Def / Thm / Proof etc.

Newly defined terms

References

Corrections

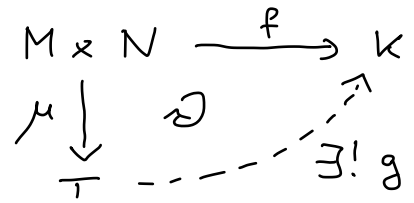
① Tensor Products of modules (Spanier: Intro, Sec 4 & 5)

Ch 5 Sec 1; Hatcher Sec 3.2 / A Künneth formula; Atiyah-MacDonald Ch 2 / Tensor Product of modules)

Let R be a commutative ring with 1 (after this section only $R = \mathbb{Z}$).

Prop 1 Let M, N be R -modules. Then there exists an R -module T and a bilinear map $\mu: M \times N \rightarrow T$ such that:

For all R -modules K and bilinear maps $f: M \times N \rightarrow K$ there is a unique homomorphism $g: T \rightarrow K$ with $g \circ \mu = f$.



Proof $U :=$ free R -module with basis the set $M \times N$.

$I :=$ submodule of U generated by

$$\begin{aligned}
 & \{ (\lambda x + x', y) - \lambda(x, y) - (x', y) \mid \lambda \in R, x, x' \in M, y \in N \} \\
 \cup & \{ (x, \lambda y + y') - \lambda(x, y) - (x, y') \mid \lambda \in R, x \in M, y, y' \in N \}
 \end{aligned}$$

Let $T = U/I$ and $\mu: M \times N \rightarrow T$, $\mu(x, y) = [(x, y)]$

Check that μ is bilinear! Now let $f: M \times N \rightarrow K$ as above be given.

Check existence of g :

Let $\tilde{g}: U \rightarrow K$ be the homomorphism with $\tilde{g}((x, y)) = f(x, y)$.

Check that $I \subseteq \ker \tilde{g} \Rightarrow \tilde{g}$ induces $g: T \rightarrow K$.

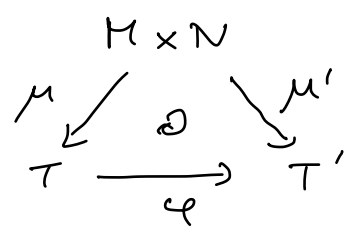
We have $g(\mu(x, y)) = g([(x, y)]) = \tilde{g}((x, y)) = f(x, y) \checkmark$

Check uniqueness of g :

If $g': T \rightarrow K$ with $g' \circ \mu = f$, then $g'([(x, y)]) = g([(x, y)])$

for all $x \in M, y \in N$. But such $[(x, y)]$ generate $T \Rightarrow g = g' \checkmark$

Prop 2 If $\mu: M \times N \rightarrow T$ and $\mu': M \times N \rightarrow T'$ both satisfy the condition in Prop 1, then there is a unique isomorphism $\varphi: T \rightarrow T'$ such that $\varphi \circ \mu = \mu'$.



Proof By assumption (existence of g), $\exists \varphi: T \rightarrow T'$ with $\varphi \circ \mu = \mu'$ and $\exists \psi: T' \rightarrow T$ with $\psi \circ \mu' = \mu$. Then $\psi \circ \varphi: T \rightarrow T$ with $\psi \circ \varphi \circ \mu = \mu$. By assumption (uniqueness of g) $\Rightarrow \psi \circ \varphi = \text{id}_T$.

Similarly $\varphi \circ \psi = \text{id}_{T'}$. □

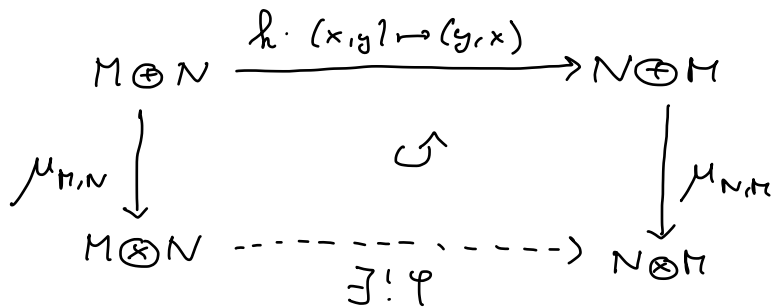
Def T as in Prop 1 is called the **tensor product of M and N over R** , written **$M \otimes_R N$** . Drop R if there is no ambiguity. Write **$x \otimes y = \mu(x, y) \in M \otimes_R N$** .

Notation \otimes and \oplus is the same for finitely many modules.

- Prop 3**
- (1) \exists iso $M \otimes N \rightarrow N \otimes M$ with $x \otimes y \mapsto y \otimes x$.
 - (2) \exists iso $(M \oplus N) \otimes K \rightarrow (M \otimes K) \oplus (N \otimes K)$ with $(x, y) \otimes z \mapsto (x \otimes z) + (y \otimes z)$
 - (3) $I \subseteq R$ ideal $\Rightarrow \exists$ iso $(R/I) \otimes M \rightarrow M/IM$ with $r \otimes m \mapsto [rm]$

Remark 4 Special case of (3): iso $R \otimes M \rightarrow M$, $r \otimes m \mapsto rm$.

Proof of Prop 3 (1)



Let $h: M \oplus N \rightarrow N \oplus M$ be the homom. with $(x,y) \mapsto (y,x)$.

Then $\mu_{N,M} \circ h: M \oplus N \rightarrow N \otimes M$ is bilinear.

By the universal property of \otimes , $\exists \varphi: M \otimes N \rightarrow N \otimes M$ with $\varphi \circ \mu_{M,N} = \mu_{N,M} \circ h$, i.e. $\varphi(x \otimes y) = y \otimes x$.

Let ψ be the analogous homo with M, N switched \Rightarrow

φ, ψ are mutually inverse homomorphisms.

In the lecture, a similar (but incorrect) proof was given, based on the erroneous assumption that h is bilinear (it is, in fact, linear).

Proof of (2) - (4): Exercises. □

Prmk 5 Using Prop 3, we can calculate $M \otimes_{\mathbb{Z}} N$ for all finitely generated abelian groups M, N .

Example 6 $\mathbb{Z}^2 \otimes \mathbb{Z}^2 = (\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}^4$

So $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ is free with basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.

Careful! Not every element of $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ is of the form $x \otimes y$,

eg $e_1 \otimes e_1 + e_2 \otimes e_2$ isn't (and isn't equal to $(e_1 + e_2) \otimes (e_1 + e_2)$).

Prmk 7 (1) Every element of $M \otimes N$ is equal to $\sum_{i=1}^n x_i \otimes y_i$ for some finite $n, x_i \in M, y_i \in N$.

(2) $(\lambda x) \otimes y = x \otimes (\lambda y)$

(3) $(x + x') \otimes y = x \otimes y + x' \otimes y$