Prop 8 $f: M \to N$, $f': M' \to N'$ R-module homoms.

(1) $\exists$ homo $f \otimes f': M \otimes M' \longrightarrow N \otimes N'$ with $x \otimes x' \mapsto f(x) \otimes f'(x')$.

(2) $(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$.

(3) $(f+g) \otimes f' = f \otimes f' + g \otimes f'$ and similarly in second factor.

Pf (1) Induced by the bilinear map $M \times M' \longrightarrow N \otimes N'$,

$$(x, x') \longmapsto f(x) \otimes f'(x').$$

(2), (3) Check that $x \otimes x'$ has the same image under both maps. $\square$

Prop 9 M an abelian group, S a commutative ring. Then $M \underset{\mathbb{Z}}{\otimes} S$ carries an S-module structure given by $s \cdot (x \otimes t) = x \otimes st$.

For homom $f: M \to N$ and S-homom $g: S \to S$,

$f \otimes g: M \otimes S \longrightarrow N \otimes S$ is an S-homom.

Proof: Exercise (careful: why is the function $x \otimes t \mapsto x \otimes st$ well-def?).

Category theory intermezzo   Weibel Sec 1.1, 1.2

Reminder A **Category** $\mathcal{C}$ consists of a class $|\mathcal{C}|$ of **objects**, for all $X, Y \in |\mathcal{C}|$ a set $\mathcal{C}(X,Y)$ of **morphisms** with a distinguished **identity morphism** $1_X \in \mathcal{C}(X,X)$, and **composition functions** $\circ: \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ such that $(f \circ g) \circ h = f \circ (g \circ h)$ and $f \circ 1_X = 1_X \circ f = f$.

A **(covariant) functor** $F: \mathcal{C} \to \mathcal{D}$ consists of functions $|\mathcal{C}| \to |\mathcal{D}|$ and $\mathcal{C}(X,Y) \longrightarrow \mathcal{D}(FX, FY)$ with $F(f \circ g) = Ff \circ Fg$ and $F 1_X = 1_{FX}$. For a **contravariant functor**, one has instead $\mathcal{C}(X,Y) \to \mathcal{D}(FY, FX)$ and $F(f \circ g) = Fg \circ Ff$.

Def A preadditive category $\mathcal{C}$ is a category with abelian group structures on $\mathcal{C}(X,Y)$, such that compositions are bilinear. A functor F between preadditive $\mathcal{C} \to \mathcal{D}$ is additive if the functions $\mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ (or $\to \mathcal{D}(FY,FX)$ if F is contravariant) are linear.

Examples 10 R commutative ring with 1.

(1) The category R-Mod of R-modules and R-homomorphisms is preadditive.

(2) Chain complex over a preadditive category $\mathcal{E}$: sequence of $C_0, C_1, \ldots \in |\mathcal{E}|$ and morphisms $d_1 : C_1 \to C_0$, $d_2 : C_2 \to C_1, \ldots$ with $d_i \circ d_{i+1} = 0$.
The cat. $Ch(\mathcal{E})$ of $\mathcal{E}$-chain complexes and chain maps is again preadditive. Chain maps $f : C \to C'$ are sequences $f_0, f_1, \ldots$ with $f_i \in \mathcal{E}(C_i, C_i')$ and $f_i \circ d_{i+1} = d'_{i+1} \circ f_{i+1}$ for all $i \geq 0$.

(3) Cat of Top spaces $\longrightarrow Ch(\mathbb{Z}\text{-Mod})$,
$X \longmapsto C(X)$, $f \longmapsto f_c$ is a functor (AlgTop I)

(3') Refinement of (3): Functor

$$\underbrace{\text{Cat of Pairs of Top spaces}} \longrightarrow Ch(\mathbb{Z}\text{-Mod})$$
$$(X,A) \longmapsto C(X,A),$$
$$f \longmapsto f_c$$

Objects: (X,A) with X Top space, $A \subseteq X$.

Morphisms $f : (X,A) \to (Y,B)$: $f : X \to Y$ cont. with $f(A) \subseteq B$.

(4) $\mathrm{Ch}(R\text{-Mod}) \longrightarrow R\text{-Mod}$, $C \longmapsto H_i(C) := \ker d_i / \operatorname{im} d_{i+1}$

$f \longmapsto f_*$ are additive functors for each fixed $i \geq 0$.

(5) Composing (3') and (4) gives functors

Pairs of top spaces $\longrightarrow \mathbb{Z}\text{-Mod}$, $(X, A) \longmapsto H_i(X, A)$,

$f \longmapsto f_*$.

(6) M a fixed R-module. Then $R\text{-Mod} \longrightarrow R\text{-Mod}$,

$N \longmapsto N \underset{R}{\otimes} M$, $\quad f \longmapsto f \otimes \mathrm{id}_M$ also written as $f \otimes M$

is an additive functor! (see Prop 8)

(6') $\mathbb{Z}\text{-Mod} \longrightarrow S\text{-Mod}$,

$M \longmapsto M \underset{\mathbb{Z}}{\otimes} S$, $\quad f \longmapsto f \otimes \mathrm{id}_S$

is another additive functor (see Prop 9)

(2) Homology with coefficients $\quad$ Spanier 5.1, Hatcher 2.2

X top. space, $A \subseteq X$, M an abelian group.

Prop 1 $\quad \cdots \xrightarrow{d_2 \otimes \mathrm{id}_M} C_1(X, A) \otimes M \xrightarrow{d_1 \otimes \mathrm{id}_M} C_0(X, A) \otimes M \longrightarrow 0$

is a chain complex.

Proof postponed.

Def We call the complex in Prop 1 the chain complex of (X, A) with coefficients in M, denoted by $C(X, A) \otimes M$. We call $H_i(C(X, A) \otimes M)$ the i-th homology group with coefficients in M, denoted by $H(X, A; M)$.

Rmk 2 $\quad C(X, A) \otimes \mathbb{Z}$ is naturally isomorphic to $C(X, A)$.

Goal Chain complexes & homology groups with any coefficients M have all the good properties proven for $\mathbb{Z}$ coefficients in Alg Top I.

Rmk 4 Recall $C_i(X)$ is a free $\mathbb{Z}$-module with basis the singular simplexes $\sigma: \Delta^i \to X \Rightarrow C_i(X) \otimes M \cong \bigoplus_{\sigma: \Delta^i \to X} M$. So one may think of a chain in $C_i(X) \otimes M$ as a finite linear combination with coefficients $m_j \in M$ of singular simplexes σ_i: $\sum_{j=1}^{k} \sigma_j \otimes m_j$.