Goal Chain complexes & homology groups with any coefficients M have all the good properties proven for $\mathbb{Z}$ coefficients in Alg Top I.

Rmk 4 Recall $C_i(X)$ is a free $\mathbb{Z}$-module with basis the singular simplexes $\sigma: \Delta^i \to X$ $\Rightarrow$ $C_i(X) \otimes M \cong \bigoplus_{\sigma: \Delta^i \to X} M$. So one may think of a chain in $C_i(X) \otimes M$ as a finite linear combination with coefficients $m_j \in M$ of singular simplexes σ_j: $\sum_{j=1}^{k} \sigma_j \otimes m_j$.

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Def (Eilenberg–Steenrod Axioms, from Alg Top I)

A homology theory is the following.

Data: For all $n \in \mathbb{Z}$:

* Functors h_n from Cat of pairs of spaces $\to$ $\mathbb{Z}$-Mod.
* Natural Homomorphisms $\partial : h_{n+1}(X, A) \to h_n(A) := h_n(A, \emptyset)$

$$\begin{tikzcd} h_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & h_n(A) \arrow[d, "f_*"] \\ h_{n+1}(Y,B) \arrow[r, "\partial"'] & h_n(B) \end{tikzcd}$$

Commutes for all Cont. $f: (X,A) \to (Y,B)$

Axioms:

(1) $f \simeq g \Rightarrow f_* = g_*$ (Homotopy)

(2) $\bar{U} \subseteq A^{\circ}$, inclusion $i: (X \setminus U, A \setminus U) \to (X, A) \Rightarrow i_*$ iso (Excision)

(3) $h_n(\text{one point space}) = 0$ for $n \neq 0$ (Dimension)

(4) For inclusions $i_\alpha: X_\alpha \to \coprod_\alpha X_\alpha$,

$$\bigoplus h_n(X_\alpha) \xrightarrow{\sum_\alpha (i_\alpha)_*} h_n\left(\coprod_\alpha X_\alpha\right) \text{ is an iso.} \quad \text{(Additivity)}$$

(5) There are long exact sequences (Exactness)

$$\cdots \to h_n(A) \xrightarrow{incl_*} h_n(X) \xrightarrow{incl_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$$

A more precise Goal **Thm 5** $H_n(\cdot\,; M)$ is a homology theory.

Prop 6 $\mathcal{F}: R\text{-Mod} \longrightarrow \mathcal{E}$ an additive functor.

(1) An additive functor $Ch(R\text{-Mod}) \longrightarrow Ch(\mathcal{E})$, which we also denote by $\mathcal{F}$, is given by sending a chain complex C,

$$\mathcal{F}(C) = \quad \cdots \longrightarrow \mathcal{F}C_2 \xrightarrow{\mathcal{F}d_2} \mathcal{F}C_1 \xrightarrow{\mathcal{F}d_1} \mathcal{F}C_0 \longrightarrow 0$$

and a chain map $f: C \longrightarrow C'$ to $\mathcal{F}(f)$ with

$\mathcal{F}(f)_i = \mathcal{F}(f_i)$.

(2) If $f, g: C \longrightarrow C'$ are homotopic, then so are $\mathcal{F}(f)$ and $\mathcal{F}(g)$.

(3) $f: C \longrightarrow C'$ a homotopy equivalence $\Rightarrow$ so is $\mathcal{F}f$.

Proof (1) $\mathcal{F}d_1 \circ \mathcal{F}d_2 = \mathcal{F}(d_1 \circ d_2) = \mathcal{F}0 = 0$ ✓

$$\begin{tikzcd} C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{i-1} \arrow[d, "f_{i-1}"] \\ C_i' \arrow[r, "d_i'"'] & C_{i-1}' \end{tikzcd} \quad \overset{\mathcal{F}}{\longmapsto} \quad \begin{tikzcd} \mathcal{F}C_i \arrow[r, "\mathcal{F}d_i"] \arrow[d, "\mathcal{F}f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}C_{i-1} \arrow[d, "\mathcal{F}f_{i-1}"] \\ \mathcal{F}C_i' \arrow[r, "\mathcal{F}d_i'"'] & \mathcal{F}C_{i-1}' \end{tikzcd} \checkmark$$

Check that $\mathcal{F}$ is an additive functor.

(2) $f \simeq g \Rightarrow \exists$ homotopy $h: C \longrightarrow C'$, ie $h_i: C_i \longrightarrow C_{i+1}'$,

$$\begin{tikzcd} C_{i+1} \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_i \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & C_{i-1} \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] \\ C_{i+1}' \arrow[r, "d'"'] & C_i' \arrow[r, "d'"'] & C_{i-1}' \end{tikzcd}$$

with $hd + d'h = f - g$

$\Rightarrow \mathcal{F}h: \mathcal{F}C \longrightarrow \mathcal{F}C'$ homotopy and $\mathcal{F}h\,\mathcal{F}d + \mathcal{F}d'\,\mathcal{F}h = \mathcal{F}f - \mathcal{F}g$.

(3) $g: C' \longrightarrow C$ and $f \circ g \simeq id_{C'}$, $g \circ f \simeq id_C \Rightarrow$

$\mathcal{F}(f) \circ \mathcal{F}(g) \simeq id_{\mathcal{F}(C')}$, $\qquad \mathcal{F}(g) \circ \mathcal{F}(f) \simeq id_{\mathcal{F}(C)}$

$\square$

Corollary 7 (apply Prop 6 to $\mathcal{F} = - \otimes M$)

(1) $C(X,A) \otimes M$ is a chain complex (that was Prop 1)

(2) Cont. $f: (X,A) \to (Y,B)$ induce chain maps

$$f_c \otimes \mathrm{id}_M : C(X,A) \otimes M \longrightarrow C(Y,B) \otimes M.$$

(3) $f \simeq g \Rightarrow f_c \otimes M \simeq g_c \otimes M$.

(4) $f_c \otimes M$ induces $f_* : H_n(X,A;M) \to H_n(Y,B;M)$

Notation We'll write f_c for $f_c \otimes \mathrm{id}_M$.

Overview of functors

$$\begin{tikzcd}
(X,A) \arrow[d, mapsto, "\substack{\text{Construction} \\ \text{in AlgTop I}}"'] & & \\
C(X,A) \arrow[r, mapsto, "-\otimes M"] \arrow[d, mapsto, "\text{homology}"'] & C(X,A)\otimes M \arrow[d, mapsto, "\text{homology}"] \\
H_n(X,A) & H_n(X,A;M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Cont. } f \arrow[d, mapsto] & \\
\text{Chain maps } f_c \arrow[r, mapsto] \arrow[d, mapsto] & f_c \arrow[d, mapsto] \\
\text{homom } f_* & f_*
\end{tikzcd}$$

Rmk 8 For a commutative ring S, $C(X,A) \otimes S$ is a chain complex over S, $H_i(X,A;S)$ is an S-module, and f_c and f_* are S-linear. Particularly useful for S a field!

We have constructed half of the data to show $H_n(- ; M)$ is a homology theory, and we have proved axiom (1) (Homotopy)

Proof of Axiom (2) (Excision) $i_c : C(X \setminus U, A \setminus U) \to C(X, A)$ is a homotopy equivalence (Alg Top I).

$- \otimes M : Ch(R\text{-Mod}) \to Ch(R\text{-Mod})$ preserves homotopy equiv. (by Prop 5(3)).

$\Rightarrow$ $i_c \otimes M$ is a hom. equiv.

$\Rightarrow$ $i_* : H_n(X \setminus U, A \setminus U ; M) \to H_n(X, A ; M)$ is an iso.

$\square$

Proof of Axiom (3) (Dimension) For X the one-point space,

$$C(X) \cong \quad \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\Rightarrow C(X) \otimes M \cong \quad \cdots \xrightarrow{id_M} M \xrightarrow{0} M \xrightarrow{id_M} M \xrightarrow{0} M \to 0$$

$$\Rightarrow H_n(X ; M) \cong \begin{cases} M & n = 0 \\ 0 & \text{else} \end{cases}$$

$\square$

Proof of Axiom (4) (Additivity) $\bigoplus_\alpha C(X_\alpha) \xrightarrow{\sum (i_\alpha)_c} C(X)$ is a homotopy equiv. (Alg Top I) $\Rightarrow$ so is $\left(\bigoplus C(X_\alpha)\right) \otimes M \xrightarrow{(\sum (i_\alpha)_c) \otimes id_M} C(X) \otimes M$,

which is isomorphic to $\bigoplus \left(C(X_\alpha) \otimes M\right) \xrightarrow{\sum (i_\alpha)_c \otimes id_M} C(X) \otimes M$ $\square$