## Construction of connecting maps $\partial$ and Proof of Axiom (5) (Exactness)

$0 \to C(A) \xrightarrow{\text{incl.}} C(X) \xrightarrow{\text{incl.}} C(X,A) \to 0$ is a SES of chain complexes of free abelian groups $\Rightarrow$

$$0 \to C(A) \otimes M \xrightarrow{\text{incl.}} C(X) \otimes M \xrightarrow{\text{incl.}} C(X,A) \otimes M \to 0$$

is also exact! (Exercise)

This concludes the proof, using:

**Lemma 8** (Alg Top I) If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ is a SES of chain complexes over a ring, then there is a LES in homology:

$$\ldots \to H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \to \ldots$$

Moreover, the $\partial$ may be chosen naturally, which means:

If
$$\begin{array}{ccccccccc} 0 & \to & C & \xrightarrow{f} & D & \xrightarrow{g} & E & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & C' & \to & D' & \to & E' & \to & 0 \end{array}$$
is commutative with exact rows

then
$$\begin{array}{ccc} H_n(E) & \xrightarrow{\partial} & H_{n-1}(C) \\ \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\alpha_*} \\ H_n(E') & \xrightarrow{\partial} & H_{n-1}(C') \end{array}$$
commutes. $\square$

Useful theorems for homology with $\mathbb{Z}$-coefficients may now be generalized to arbitrary coefficients $M$ in one of the following ways:

* Deduce from Eilenberg–Steenrod axioms
* Deduce from the $\mathbb{Z}$-version
* Prove in the same way as for $\mathbb{Z}$

**Prop 9** $H_0(X;M) \cong \bigoplus_{Z \in \pi_0(X)} \underbrace{\{[\sigma_Z \otimes m] \mid m \in M\}}_{\cong M}$, where one chooses

$\sigma_Z : \underset{\{*\}}{\overset{}{\Delta^0}} \to X$, $\sigma(*) \in Z$ for each path-connected comp. $Z \in \pi_0(X)$.

**Theorem 10** (Mayer-Vietoris) If $A, B \subseteq X$ with $A^\circ \cup B^\circ = X$, then there is a LES

$$\cdots \to H_n(A\cap B; M) \xrightarrow{\binom{incl_*}{incl_*}} H_n(A;M) \oplus H_n(B;M) \xrightarrow{(incl_* \; -incl_*)} H_n(X;M) \to H_{n-1}(A\cap B; M) \to \cdots$$

**Theorem 11** If $(X, A)$ is a good pair (ie $A \subseteq X$ is closed and a strong deformation retract of $X$), then the projection map $p: X \to X/A$ induces isos

$$p_*: H_n(X, A; M) \to H_n(X/A, A/A; M) \cong \tilde{H}_n(X/A; M)$$

**Remark 12** Reduced homology groups $\tilde{H}_n(X;M)$ may be defined as over $\mathbb{Z}$ coefficients for $X \neq \emptyset$. One has

$$\tilde{H}_n(X;M) \cong H_n(X, \{x_0\}; M) \underset{\text{if } n>0}{\cong} H_n(X)$$

and $H_0(X;M) \cong M \oplus \tilde{H}_0(X;M)$.

**Def** (AlgTopI) $X$ a CW-complex with cells $e^n_\alpha$. Let

$C^{CW}_n(X)$ = free abelian group with basis $e^n_\alpha$ and

$d: C^{CW}_n(X) \to C^{CW}_{n-1}(X)$ given by $d e^n_\alpha = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$,

where $d_{\alpha\beta} \in \mathbb{Z}$ is the degree of

$$S^{n-1} \xrightarrow[\text{attaching map of } e^n_\alpha]{} X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta) \cong S^{n-1}$$

($X^{n-1}$: $(n-1)$-skeleton of $X = \bigcup_{k \leq n, \alpha} e^k_\alpha$)

$C^{CW}(X)$ is the cellular chain complex of $X$ and

$H^{CW}_n(X) := H_n(C^{CW}(X))$ the cellular homology of $X$.

**Theorem 13** $H^{CW}_n(X;M) := H_n(C^{CW}(X) \otimes M) \cong H_n(X;M)$

## ③ Calculations & the theorem of Borsuk–Ulam

**Prop 1** For all $k \geq 0$, $\tilde{H}_n(S^k; M) \cong M$ if $n = k$, trivial otherwise.

**Three ways to prove it** (1) $S^k$ has a CW structure with one 0-cell, one k-cell.

(2) Mayer-Vietoris with $A = S^k \setminus e_1$, $B = S^k \setminus -e_1$

(3) LES of the good pair $(D^k, \partial D^k)$ □

**Def** Real Projective k-space $\mathbb{R}P^k := S^k / x \sim -x$

**Rmk 2** * $\mathbb{R}P^k \cong (\mathbb{R}P^{k+1} \setminus \vec{0}) / x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus 0$

* $\mathbb{R}P^0$ = one point space, $\mathbb{R}P^1 \cong S^1$

* Alg Top I: $$H_n(\mathbb{R}P^k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & 1 \leq n \leq k-1,\ n \text{ odd} \\ 0 & 1 \leq n \leq k-1,\ n \text{ even} \\ \mathbb{Z} & n = k \text{ odd} \\ 0 & n = k \text{ even} \\ 0 & k+1 \leq n \end{cases}$$

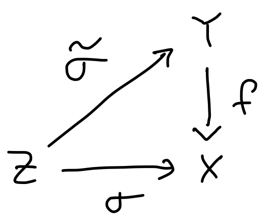
**Prop 3** $H_n(\mathbb{R}P^k; \mathbb{Z}/2) \cong \mathbb{Z}/2$ if $0 \leq n \leq k$ and $0$ otherwise.

**Prop 4** Let $f: Y \to X$ be a twofold covering. Then there is a LES

$$\cdots \to H_n(X; \mathbb{Z}/2) \to H_n(Y; \mathbb{Z}/2) \xrightarrow{f_*} H_n(X; \mathbb{Z}/2) \to H_{n-1}(X; \mathbb{Z}/2) \to \cdots$$

(a special case of the Gysin LES)

**Proof** Recall that: a cont. map $\sigma: Z \to X$ on a contractible space $Z$ has exactly two lifts $\tilde{\sigma}_1, \tilde{\sigma}_2 : Z \to Y$. Here, a lift is a map $\tilde{\sigma}: Z \to Y$ so that




commutes.

Define the so-called transfer homomorphism $T: C_n(X) \to C_n(Y)$ by $T(\sigma: \Delta^n \to X) = \tilde{\sigma}_1 + \tilde{\sigma}_2$. Check that $T$ is a chain map.

We'll show that the short sequence of complexes

$$0 \to C(X) \otimes \mathbb{Z}/2 \xrightarrow{T} C(Y) \otimes \mathbb{Z}/2 \xrightarrow{f_c} C(X) \otimes \mathbb{Z}/2 \to 0$$

is exact. This induces the desired LES in homology (Lemma 2.9).

* $f_c$ surjective: Lifts exist. ✓

* $T$ is injective. For a sing simplex $\tau: \Delta^n \to X$,

  let $p_\tau : C(X) \otimes \mathbb{Z}/2 \to \mathbb{Z}/2$ be the projection $\sum_\sigma \sigma \otimes \lambda_\sigma \mapsto \lambda_\tau$.

  $c = \sum_\sigma \sigma \otimes \lambda_\sigma \neq 0 \Rightarrow \exists\, \tau$ with $\lambda_\tau = 1$ for some $\tau$

  $\Rightarrow \lambda_{\tilde{\tau}}(T(c)) = 1$ for $\tilde{\tau}$ a lift of $\tau \Rightarrow T(c) \neq 0$. ✓

* $\operatorname{im}(T) = \ker f_c$. $f_c(c = \sum_\sigma \sigma \otimes \lambda_\sigma) = 0$

  $\Leftrightarrow p_\tau(f_c(c)) = 0 \quad \forall \tau: \Delta^n \to X.$

  Since $p_\tau(f_c(c)) = p_{\tilde{\tau}_1}(c) + p_{\tilde{\tau}_2}(c)$, it follows that

  $$f_c(c) = 0 \Leftrightarrow c = \sum_{\tau: \Delta^n \to X} \lambda_\tau (\tilde{\tau}_1 + \tilde{\tau}_2) = T\Big(\sum_\tau \lambda_\tau \tau\Big)$$

  $\Leftrightarrow c \in \operatorname{im}(T)$. ✓

□