Prop 4 Let $f: Y \to X$ be a twofold covering. Then there is a LES

$$\cdots \to H_n(X;\mathbb{Z}/2) \to H_n(Y;\mathbb{Z}/2) \xrightarrow{f_*} H_n(X;\mathbb{Z}/2) \to H_{n-1}(X;\mathbb{Z}/2) \to \cdots$$

(a special case of the Gysin LES)

Today For the remainder of ③: $H_n(X,A)$ means $H_n(X,A;\mathbb{Z}/2)$

Prop 3 $H_n(\mathbb{R}P^k) \cong \mathbb{Z}/2$ if $0 \le n \le k$ and 0 otherwise.

Proof We already know this for $n=0,1$. So assume $n \ge 2$.

For the covering $f: S^n \to \mathbb{R}P^n$, the Gysin LES breaks into pieces:

$$\boxed{0 \to H_1(\mathbb{R}P^n) \xrightarrow{\partial} H_0(\mathbb{R}P^n) \xrightarrow{T_*} H_0(S^n) \xrightarrow{f_*} H_0(\mathbb{R}P^n) \to 0}$$

All homology groups are $\mathbb{Z}/2$ - vector spaces (by Rmk 2.8).

f_* surjective and $H_0(S^n) \Rightarrow H_0(\mathbb{R}P^n) \cong \mathbb{Z}/2$ or 0.

Exactness at $H_0(S^n) \Rightarrow H_0(\mathbb{R}P^n) \cong \mathbb{Z}/2 \Rightarrow f_* = 1 \Rightarrow T_* = 0$

$\Rightarrow H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$.

$$\boxed{0 \to H_k(\mathbb{R}P^n) \xrightarrow{\partial} H_{k-1}(\mathbb{R}P^n) \to 0} \quad \text{if } k \notin \{0,1,n,n+1\}$$

So, $H_k(\mathbb{R}P^n) \cong H_{k-1}(\mathbb{R}P^n) \Rightarrow H_k(\mathbb{R}P^n) \cong \mathbb{Z}/2$ for $k \le n-1$

by induction.

$$\boxed{0 \to H_{n+1}(\mathbb{R}P^n) \xrightarrow{\partial} H_n(\mathbb{R}P^n) \xrightarrow{T_*} \underbrace{H_n(S^n)}_{\mathbb{Z}/2} \xrightarrow{f_*} H_n(\mathbb{R}P^n) \xrightarrow{\partial} \underbrace{H_{n-1}(\mathbb{R}P^n)}_{\mathbb{Z}/2} \to 0}$$

Since $\mathbb{R}P^n$ has a CW-structure without k-cells for $k \ge n+1$

$\Rightarrow H_k(\mathbb{R}P^n) = 0$ for $k \ge n+1$.

$\Rightarrow H_n(\mathbb{R}P^n)$ surjects onto $\mathbb{Z}/2$, and injects into $\mathbb{Z}/2$

$\Rightarrow H_n(\mathbb{R}P^n) \cong \mathbb{Z}/2$. $\square$

Prop 5 The Gysin sequence from Prop 4 is natural, ie if

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[d, "\beta"] \\ Y' \arrow[r, "f'"'] & X' \end{tikzcd}$$

commutes and f, f' are twofold coverings, then

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X) \arrow[r, "T_*"] \arrow[d, "\beta_*"] & H_n(Y) \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & H_n(X) \arrow[r, "\partial"] \arrow[d, "\beta_*"] & H_{n-1}(X) \arrow[r] \arrow[d, "\beta_*"] & \cdots \\ \cdots \arrow[r] & H_n(X') \arrow[r, "T_*"'] & H_n(Y') \arrow[r, "f'_*"'] & H_n(X') \arrow[r, "\partial"'] & H_{n-1}(X') \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Proof Check that

$$\begin{tikzcd} 0 \arrow[r] & C_n(X)\otimes \mathbb{Z}/2 \arrow[r, "T"] \arrow[d, "\beta_c"'] & C_n(Y)\otimes \mathbb{Z}/2 \arrow[r, "f_c"] \arrow[d, "\alpha_c"'] & C_n(X)\otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "\beta_c"'] & 0 \\ 0 \arrow[r] & C_n(X')\otimes \mathbb{Z}/2 \arrow[r, "T"'] & C_n(Y')\otimes \mathbb{Z}/2 \arrow[r, "f'_c"'] & C_n(X')\otimes \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

commutes, then use Lemma 2.8. $\square$

Borsuk-Ulam Theorem $f: S^n \to \mathbb{R}^n$ continuous $\Rightarrow$ $\exists x \in S^n: f(x) = f(-x)$.

Proof If no such x exist, let $g: S^n \longrightarrow S^{n-1}$,

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x)\|}.$$

Then $g(-x) = -g(x)$.

This contradicts the following theorem. $\square$

Theorem 6 There is a cont. map $g: S^n \to S^m$ with $g(-x) = -g(x)$ $\iff$ $n \leq m$.

Proof If $n \leq m$, the embedding $i: (x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_{n+1}, 0, \dots 0)$ satisfies $i(-x) = -i(x)$.

For the other direction, assume $n > m \geq 1$ and

Let such a g be given. If $p_n(x) = p_n(y)$, then $p_m \circ g(x) = p_m \circ g(y)$.
Because the covering p_n is a quotient map, there is $h: \mathbb{R}P^n \to \mathbb{R}P^m$ s.t.

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "p_n"'] \arrow[dr, "p_m \circ g"] & S^m \arrow[d, "p_m"] \\
\mathbb{R}P^n \arrow[r, "h"'] & \mathbb{R}P^m
\end{tikzcd}$$

commutes.

Now, apply Prop 5 (naturality of the Gysin sequence) to the pieces of the Gysin LES (see proof of Prop 3):

$$\begin{tikzcd}
0 \arrow[r] & H_k(\mathbb{R}P^n) \arrow[r, "\text{iso}"] \arrow[d, "h_{*,k}"] & H_{k-1}(\mathbb{R}P^n) \arrow[r] \arrow[d, "h_{*,k-1}"] & 0 \\
0 \arrow[r] & H_k(\mathbb{R}P^m) \arrow[r, "\text{iso}"'] & H_{k-1}(\mathbb{R}P^m) \arrow[r] & 0
\end{tikzcd}$$

commute for $1 \leq k \leq m-1$. Also, $h_{*,0}$ iso because $\mathbb{R}P^n, \mathbb{R}P^m$ path-connected $\Rightarrow h_{*,1}$ iso $\Rightarrow h_{*,2}$ iso $\Rightarrow \dots \Rightarrow h_{*,m-1}$ iso.

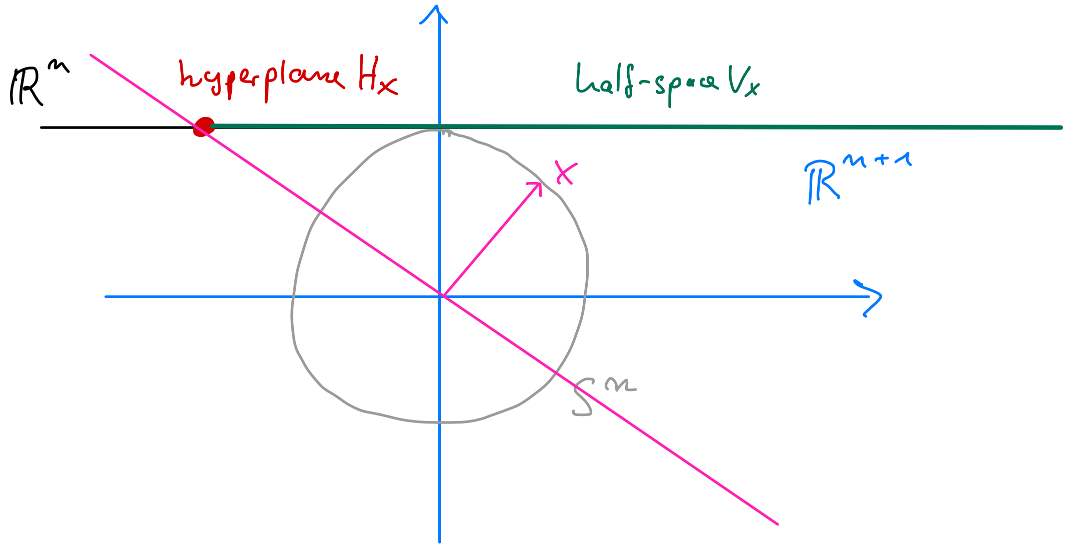
$$\begin{tikzcd}
{} \arrow[r, "\text{iso}"] & \overset{\mathbb{Z}/2}{H_m(\mathbb{R}P^n)} \arrow[r, "0"] \arrow[d, "\text{iso}"] \arrow[dr, phantom, "\text{does not commute!}"] & \overset{0}{H_m(S^n)} \arrow[r, "0"] \arrow[d] & \overset{\mathbb{Z}/2}{H_m(\mathbb{R}P^n)} \arrow[r, "\text{iso}"] \arrow[d, "\text{iso}"] & \overset{\mathbb{Z}/2}{H_{m-1}(\mathbb{R}P^n)} \arrow[r] \arrow[d, "\text{iso}"] & 0 \\
0 \arrow[r] & \underset{\mathbb{Z}/2}{H_m(\mathbb{R}P^m)} \arrow[r, "\text{iso}"'] & \underset{\mathbb{Z}/2}{H_m(S^m)} \arrow[r, "0"'] & \underset{\mathbb{Z}/2}{H_m(\mathbb{R}P^m)} \arrow[r, "\text{iso}"'] & \underset{\mathbb{Z}/2}{H_{m-1}(\mathbb{R}P^m)} \arrow[r] & 0
\end{tikzcd}$$

Contradiction! $\square$

The Ham Sandwich Theorem $A_1, \dots, A_n \subseteq \mathbb{R}^n$ Lebesgue-measurable & bounded

$\Rightarrow \exists$ hyperplane in $\mathbb{R}^n$ cutting each A_i in half by volume.

Proof Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$.



For $x \in S^n$, let
$$H_x = \mathbb{R}^n \times \{1\} \cap \{y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle = 0\}$$
$$V_x = \mathbb{R}^n \times \{1\} \cap \{y \in \mathbb{R}^{n+1} \mid \langle x, y \rangle \geq 0\}$$

Let $f : S^n \to \mathbb{R}^n$, $f_i(x) = \mathrm{vol}(V_x \cap A_i)$.

f is continuous since the A_i are bounded.

Borsuk-Ulam $\Rightarrow \exists\, x \in S^n : f(x) = f(-x)$

$\Rightarrow \mathrm{vol}(V_x \cap A_i) = \mathrm{vol}(V_{-x} \cap A_i) = \mathrm{vol}(A_i \setminus V_x)$

$\Rightarrow H_x$ cuts all A_i in half.

$\square$