4 The Universal Coefficient Theorem for Homology

8 March

The splitting Lemma For a SES $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ of abelian groups, the following are equivalent:

(1) There is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "id_M"'] & N \arrow[r, "g"] \arrow[d, "iso"] & P \arrow[r] \arrow[d, "id_P"] & 0 \\ 0 \arrow[r] & M \arrow[r, "incl"'] & M \oplus P \arrow[r, "proj"'] & P \arrow[r] & 0 \end{tikzcd}$$

(2) $\exists\, i: P \to N$ with $g \circ i = id_P$.

(3) $\exists\, r: N \to M$ with $r \circ f = id_M$

SES satisfying these conditions are called Split.

UCT for Homology Let C be a chain complex of free abelian groups. Let M be an abelian group.

(1) For all n, there is a split SES of abelian groups:

$$0 \to H_n(C) \otimes M \xrightarrow{[x]\otimes m \mapsto [x \otimes m]} H_n(C; M) \to \operatorname{Tor}(H_{n-1}(C), M) \to 0$$

(2) This SES is natural, ie for a chain map $f: C \to C'$

$$\begin{tikzcd} 0 \arrow[r] & H_n(C) \otimes M \arrow[r] \arrow[d, "f_* \otimes id_M"] & H_n(C; M) \arrow[r] \arrow[d, "f_*"] & \operatorname{Tor}(H_{n-1}(C), M) \arrow[r] \arrow[d, "\operatorname{Tor}(f_*, id_M)"] & 0 \\ 0 \arrow[r] & H_n(C') \otimes M \arrow[r] & H_n(C'; M) \arrow[r] & \operatorname{Tor}(H_{n-1}(C'), M) \arrow[r] & 0 \end{tikzcd}$$

commutes.

(3) There is no natural choice of splitting maps
$\to$ Exercise 2.4

Correction 12 March

In the lecture it was erroneously claimed that "or" suffices here

Remark 1 $\operatorname{Tor}(N, M)$ will be defined for all abelian groups N, M.

We will show that ~~for~~ if M and N are finitely generated, then

$\operatorname{Tor}(N, M) \cong T(N) \otimes T(M)$, where

$T(N) = \{x \in N \mid \exists \lambda \in \mathbb{Z} \setminus \{0\} : \lambda x = 0\}$ is the torsion subgroup of N.

Remark The UCT implies that homology with any coefficients can be read off homology with $\mathbb{Z}$ coefficients, i.e. $\mathbb{Z}$ coefficients are "universal". However, for a cont. map f, f_* on $H(-;M)$ is in general not determined by f_* on $H(-;\mathbb{Z})$.
→ Exercise 2.4

Example 2 For $\mathbb{RP}^3$, $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}/2$, $H_2 \cong 0$, $H_3 = \mathbb{Z}$

UCT for $M = \mathbb{Z}/2$:

$$0 \to \underbrace{H_1(\mathbb{RP}^3)\otimes \mathbb{Z}/2}_{\mathbb{Z}/2} \to \underbrace{H_1(\mathbb{RP}^3;\mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\mathrm{Tor}(\overbrace{H_0(\mathbb{RP}^3)}^{\mathbb{Z}},\mathbb{Z}/2)}_{0} \to 0$$

$$0 \to \underbrace{H_2(\mathbb{RP}^3)\otimes \mathbb{Z}/2}_{0} \to \underbrace{H_2(\mathbb{RP}^3;\mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\mathrm{Tor}(\overbrace{H_1(\mathbb{RP}^3)}^{\mathbb{Z}/2},\mathbb{Z}/2)}_{\mathbb{Z}/2} \to 0$$

Reminder M finitely generated abelian group $\Rightarrow$

$$M = M^a \oplus \bigoplus_{\substack{p \text{ prime} \\ r \geq 1}} (\mathbb{Z}/_{p^r})^{b_{p,r}}$$ with $a, b_{p,r}$ uniquely determined.

a is called the rank of M, written $\mathrm{rk}\, M$ or $\mathrm{rank}\, M$.

Prop 3 Assume $\bigoplus_n H_n(X)$ is finitely generated. Let $\mathbb{F}$ be a field of characteristic p.

$$\dim_{\mathbb{F}} H_n(X;\mathbb{F}) = \begin{cases} \mathrm{rank}\, H_n(X) & \text{if } p = 0 \\ \mathrm{rank}\, H_n(X) \\ \quad + \#\, \mathbb{Z}/p^r\text{-summands of } H_n(X) & \text{else} \\ \quad + \#\, \mathbb{Z}/p^r\text{-summands of } H_{n-1}(X) \end{cases}$$

Proof UCT $\Rightarrow$ $H_n(X;\mathbb{F}) \cong H_n(X)\otimes\mathbb{F} \oplus \underbrace{\mathrm{Tor}(H_{n-1}(X),\mathbb{F})}$

$\cong T(H_{n-1}(X)) \otimes T(\mathbb{F})$ by Remark 1

Correction 12 March
The Proposition is true, but the proof doesn't work in general since $\mathbb{F}$ need not be finitely generated. We'll need to understand Tor better first to prove Prop 3

Now use $T(\mathbb{F}) = \begin{cases} 0 & \text{if } p = 0 \\ \mathbb{F} & \text{else} \end{cases}$

and $\mathbb{Z}/m \otimes \mathbb{F} \cong \mathbb{F}/m \cong \begin{cases} 0 & p \mid m \\ \mathbb{F} & \text{else} \end{cases}$ $\square$

Prop 4 Let X be a space s.t. $H_n(X) \cong 0$ for sufficiently large n, and $H_n(X)$ finitely generated for all n. Then

$$\sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{F}}(H_n(X;\mathbb{F})) \in \mathbb{Z}$$

does not depend on the choice of a field $\mathbb{F}$. This integer is called the Euler characteristic of X, written $\chi(X)$.

Proof Note that (# $\mathbb{Z}/p^r$-summands of $H_n(X)$) appears as summand in $\dim H_n(X;\mathbb{F})$ and in $\dim H_{n+1}(X;\mathbb{F})$. So, this cancels in χ due to opposite signs. $\square$

To prove the UCT, we need a fundamental tool of homological algebra. Let R be a commutative ring.

Def A free resolution of an R-Module M is a LES

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where the F_i are free R-Modules.