Last time

To prove the UCT, we need a fundamental tool of homological algebra. Let R be a commutative ring.

Def A **free resolution** $\mathcal{F}$ of an R-Module M is a LES

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where the F_i are free R-Modules.

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Note that $\cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$ is a chain complex. It is called **deleted resolution**, denoted $\mathcal{F}^M$, with $H_0(\mathcal{F}^M) \cong M$, $H_n(\mathcal{F}^M) \cong 0$ for $n \neq 0$. Understanding $H_n(\mathcal{F}^M; N)$ is a special case of understanding $H_n(C; N)$ for all complexes!

Ex For $R = \mathbb{Z}$:

$$\begin{array}{l}
\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \underline{M}: \mathbb{Z}/3 \to 0 \\
\qquad\qquad\qquad \cdots\ 0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \\
\qquad\qquad \cdots\ 0 \to \mathbb{Z} \xrightarrow{\binom{1}{-1}} \mathbb{Z}^2 \xrightarrow{(1\ \ 1)} \mathbb{Z} \to 0 \\
\qquad\qquad\qquad\qquad ? \to \mathbb{Q} \to 0
\end{array}$$

Prop 5 Every module has a free resolution.

Lemma 6 For every module M there exists a free module $\mathcal{F}$ with a surjection $p: \mathcal{F} \to M$.

Proof $\mathcal{F} := \bigoplus_{x \in M} R_x$ with $R_x \cong R$. $\mathcal{F}$ is free (with basis indexed by M) and $p: \mathcal{F} \to M$, $R_x \ni 1 \mapsto x$ is surjective. $\square$

Proof of Prop 5 Pick $d_0 : F_0 \to M$ with d_0 surjective, F_0 free.
Pick $d_1' : F_1 \to \ker d_0$ with d_1' surjective, F_1 free and let
$d_1 : F_1 \to F_0$, $d_1 = (\ker d_0 \hookrightarrow F_0) \circ d_1'$.
Pick $d_2' : F_2 \to \ker d_1$ with d_2' surjective, F_2 free ... etc. □

Thm 7 Every subgroup of a free abelian group is free abelian.
Proof using Zorn's Lemma (see eg Lang "Algebra" Appendix 2 §2)

Prop 8 For $R = \mathbb{Z}$: Every abelian group M has a free resolution of length 1, ie $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$

Proof Pick $d_0 : F_0 \to M$ with d_0 surjective, F_0 free. By Thm ,
$\ker d_0$ is free. So let $F_1 = \ker d_0$, and d_1 the inclusion. □

Prop 9 ("Comparison Thm", "Fundamental Thm of Homological Algebra")

(1) If $f : M \to N$ is R-linear and F, G are free resolutions of M, N, then f may be extended to a chain map $\hat{f} : F^M \to G^N$, ie

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[d, dashed, "\exists \hat{f}_1"] & F_0 \arrow[r, "d_0"] \arrow[d, dashed, "\exists \hat{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & G_1 \arrow[r, "e_1"'] & G_0 \arrow[r, "e_0"'] & N \arrow[r] & 0
\end{tikzcd}$$

(2) $\hat{f}$ is unique up to homotopy.

(3) F, G free resolutions of M $\Rightarrow$ The unique chain map $F^M \to G^M$ extending id_M is a homotopy equivalence.

Proof (1)

$$\begin{tikzcd} \mathcal{F}_0 \arrow[d, dotted, "\exists \hat{f}_0"'] \arrow[dr, "f\circ d_0"] & \\ G_0 \arrow[r, "e_0"'] & N \end{tikzcd}$$

Since e_0 surjective and $\mathcal{F}_0$ free, there is $\hat{f}_0 : \mathcal{F}_0 \to G_0$ making the diagram commute (proof: for each basis element b of $\mathcal{F}_0$, pick $\hat{f}_0(b)$ such that $e_0(\hat{f}_0(b)) = f(d_0(b))$.

$$\begin{tikzcd} \mathcal{F}_1 \arrow[dr, "\hat{f}_0\circ d_1"] & \\ G_1 \arrow[r, "e_1"'] & G_0 \end{tikzcd}$$

$f(d_0(d_1(x))) = 0 \quad \forall x \quad \Rightarrow e_0(\hat{f}_0(d_1(x))) = 0 \quad \forall x$

$\Rightarrow \operatorname{im} \hat{f}_0 \circ d_1 \subseteq \ker e_0 = \operatorname{im} e_1.$

$\Rightarrow \exists \hat{f}_1 : \mathcal{F}_1 \to G_1$ making the diagr commute etc.

(2) Let two such chain maps be given, and let g be their difference. Then:

$$\begin{tikzcd} \cdots \mathcal{F}_2 \arrow[r, "d_2"] \arrow[d, "g_2"'] & \mathcal{F}_1 \arrow[r, "d_1"] \arrow[d, "g_1"] \arrow[dl, "h_1"] & \mathcal{F}_0 \arrow[r, "d_0"] \arrow[d, "g_0"] \arrow[dl, "h_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ \cdots G_2 \arrow[r, "e_2"'] & G_1 \arrow[r, "e_1"'] & G_0 \arrow[r, "e_0"'] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. $0 = 0 \circ d_0 = e_0 \circ g_0 \Rightarrow \operatorname{im} g_0 \subseteq \ker e_0 = \operatorname{im} e_1.$

$\Rightarrow \exists\, h_0$ with $e_1 \circ h_0 = g_0$

$$e_1 \circ (g_1 - h_0 \circ d_1) = e_1 \circ g_1 - g_0 \circ d_1 = 0$$

$\Rightarrow \exists\, h_1$ with $e_2 \circ h_1 = g_1 - h_0 \circ d_1$ etc.

(3) $\mathcal{F}, G$ free res. of M $\Rightarrow \exists$ chain maps $\hat{f} : \mathcal{F}^M \to G^M$ and $\hat{g} : G^M \to \mathcal{F}^M$ that extend $\mathrm{id}_M : M \to M$ $\Rightarrow \hat{g}\circ\hat{f} : \mathcal{F}^M \to \mathcal{F}^M$ and $\hat{f}\circ\hat{g} : G^M \to G^M$ extend id_M, but so do $\mathrm{id}_{\mathcal{F}^M}$, id_{G^M}

$\Rightarrow$ By uniqueness, $\hat{g}\circ\hat{f} \simeq \mathrm{id}_{\mathcal{F}^M}$, $\hat{f}\circ\hat{g} \simeq \mathrm{id}_{G^M}$. $\square$

Def Let M, N be R-Modules, and $\mathcal{F}$ a free resolution of M, then $\mathrm{Tor}_n(M,N) := H_n(\mathcal{F}^M; N)$ for $n \geq 0$.

Proof that Tor does not depend on choice of $\mathcal{F}$: $\mathcal{F}, \mathcal{G}$ free res. of M $\Rightarrow \mathcal{F}^M \simeq \mathcal{G}^M \Rightarrow \mathcal{F}^M \otimes N \simeq \mathcal{G}^M \otimes N$ (Cor 2.7 (3)) $\Rightarrow$ $H_n(\mathcal{F}^M; N) \cong H_n(\mathcal{G}^M; N)$. $\square$

Remark 10 Over $R = \mathbb{Z}$, $\mathrm{Tor}_n(M,N) = 0$ $\forall n \geq 2$ since M has a free res. of length 1 (Prop 8). So we write $\mathrm{Tor}(M,N) := \mathrm{Tor}_1(M,N)$.

Lemma 11 $f: M \to N$ R-linear, P R-module $\Rightarrow$ $(\mathrm{Coker}\, f) \otimes P \cong \mathrm{Coker}(f \otimes id_P)$. **Proof** Exercise.

Proof of the UCT (1) Constructing the SES

$$\underbrace{B_n = \mathrm{im}\, d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \ker d_n}_{n\text{-cycles}}$$

Make B_n, Z_n into chain complexes, taking 0 as differential. There is a SES of chain complexes:

$$\begin{tikzcd}
 & & \vdots & & \\
0 \arrow[r] & Z_{n+1} \arrow[r, "incl"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d"] \arrow[d, "d"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "incl"] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd}$$