

To prove the UCT, we need a fundamental tool of homological algebra. Let  $R$  be a commutative ring.

**Def** A **free resolution**  $F$  of an  $R$ -Module  $M$  is a LES

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

where the  $F_i$  are free  $R$ -Modules.

**Today**

13 March

Note that  $\dots \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow 0$  is a chain complex. It is called **deleted resolution**, denoted  $F^M$ , with  $H_0(F^M) \cong M$ ,  $H_n(F^M) \cong 0$  for  $n \neq 0$ . Understanding  $H_n(F^M; N)$  is a special case of understanding  $H_n(C; N)$  for all complexes!

$$\begin{array}{ccccccc} \text{Ex} & \text{For } R = \mathbb{Z}: & \dots \rightarrow 0 & \rightarrow 0 & \rightarrow \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} \rightarrow \mathbb{Z}/3 \rightarrow 0 \\ & & & & & & \dots 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \\ & & & & & & \dots 0 \rightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z} \rightarrow 0 \\ & & & & & & \quad \quad \quad ? \rightarrow \mathbb{Q} \rightarrow 0 \end{array}$$

**Prop 5** Every module has a free resolution.

**Lemma 6** For every module  $M$  there exists a free module  $F$  with a surjection  $p: F \rightarrow M$ .

**Proof**  $F := \bigoplus_{x \in M} R_x$  with  $R_x \cong R$ .  $F$  is free (with basis

indexed by  $M$ ) and  $p: F \rightarrow M$ ,  $R_x \ni 1 \mapsto x$  is surjective.  $\square$

**Proof of Prop 5** Pick  $d_0: F_0 \rightarrow M$  with  $d_0$  surjective,  $F_0$  free.

Pick  $d'_1: F_1 \rightarrow \ker d_0$  with  $d'_1$  surjective,  $F_1$  free and let

$$d_1: F_1 \rightarrow F_0, \quad d_1 = (\ker d_0 \hookrightarrow F_0) \circ d'_1.$$

Pick  $d'_2: F_2 \rightarrow \ker d_1$  with  $d'_2$  surjective,  $F_2$  free ... etc.  $\square$

**Thm 7** Every subgroup of a free abelian group is free abelian.

Proof using Zorn's Lemma (see eg Lang "Algebra" Appendix 2 §2)

**Prop 8** For  $R = \mathbb{Z}$ : Every abelian group  $M$  has a free resolution of

length 1, ie 
$$0 \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

**Proof** Pick  $d_0: F_0 \rightarrow M$  with  $d_0$  surjective,  $F_0$  free. By Thm ,

$\ker d_0$  is free. So let  $F_1 = \ker d_0$ , and  $d_1$  the inclusion.  $\square$

**Prop 9** ("Comparison Thm", "Fundamental Thm of Homological Algebra")

(1) If  $f: M \rightarrow N$  is  $R$ -linear and  $F, G$  are free resolutions of  $M, N$ ,

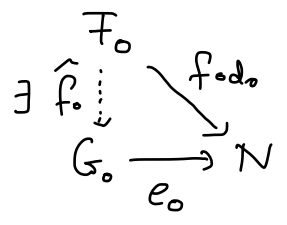
then  $f$  may be extended to a chain map  $\hat{f}: F^M \rightarrow G^N$ , ie

$$\begin{array}{ccccccc} \dots & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \rightarrow 0 \\ & \downarrow \exists \hat{f}_1 & & \downarrow \exists \hat{f}_0 & & \downarrow f & \\ \dots & G_1 & \xrightarrow{e_1} & G_0 & \xrightarrow{e_0} & N & \rightarrow 0 \end{array}$$

(2)  $\hat{f}$  is unique up to homotopy.

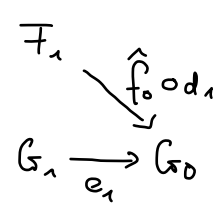
(3)  $F, G$  free resolutions of  $M \Rightarrow$  The unique chain map  $F^M \rightarrow G^M$  extending  $\text{id}_M$  is a homotopy equivalence.

Proof (1)



Since  $e_0$  surjective and  $F_0$  free, there is  $\hat{f}_0 : F_0 \rightarrow G_0$  making the diagram commute (proof: for each basis element

$b$  of  $F_0$ , pick  $\hat{f}_0(b)$  such that  $e_0(\hat{f}_0(b)) = f(d_0(b))$ .

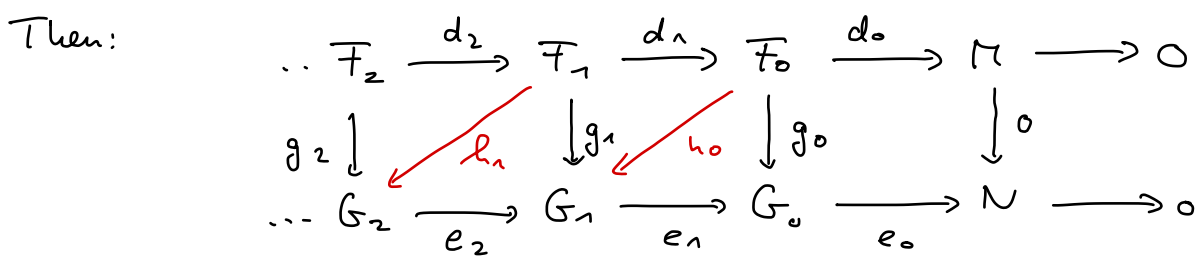


$$f(d_0(d_1(x))) = 0 \quad \forall x \Rightarrow e_0(\hat{f}_0(d_1(x))) = 0 \quad \forall x$$

$$\Rightarrow \text{im } \hat{f}_0 \circ d_1 \subseteq \ker e_0 = \text{im } e_1$$

$\Rightarrow \exists \hat{f}_1 : F_1 \rightarrow G_1$  making the diagr commute etc.

(2) Let two such chain maps be given, and let  $g$  be their difference.



Commutates.  $0 = 0 \circ d_0 = e_0 \circ g_0 \Rightarrow \text{im } g_0 \subseteq \ker e_0 = \text{im } e_1$ .

$\Rightarrow \exists h_0$  with  $e_1 \circ h_0 = g_0$

$$e_1 \circ (g_1 - h_0 \circ d_1) = e_1 \circ g_1 - g_0 \circ d_1 = 0$$

$\Rightarrow \exists h_1$  with  $e_2 \circ h_1 = g_1 - h_0 \circ d_1$  etc.

(3)  $F, G$  free res. of  $M \Rightarrow \exists$  chain maps  $\hat{f} : F^M \rightarrow G^M$  and

$\hat{g} : G^M \rightarrow F^M$  that extend  $\text{id}_M : M \rightarrow M \Rightarrow \hat{g} \circ \hat{f} : F^M \rightarrow F^M$

and  $\hat{f} \circ \hat{g} : G^M \rightarrow G^M$  extend  $\text{id}_M$ , but so do  $\text{id}_{F^M}, \text{id}_{G^M}$

$\Rightarrow$  By uniqueness,  $\hat{g} \circ \hat{f} \simeq \text{id}_{F^M}, \hat{f} \circ \hat{g} \simeq \text{id}_{G^M}$ . □

**Def** Let  $M, N$  be  $R$ -Modules, and  $F$  a free resolution of  $M$ , then  $\text{Tor}_n(M, N) := H_n(F^n; N)$  for  $n \geq 0$ .

**Proof that Tor does not depend on choice of  $F$ :**  $F, G$  free res. of  $M$   
 $\Rightarrow F^n \cong G^n \Rightarrow F^n \otimes N \cong G^n \otimes N$  (Cor (2) 7 (3))  $\Rightarrow$   
 $H_n(F^n; N) \cong H_n(G^n; N)$ . □

**Remark 10** Over  $R = \mathbb{Z}$ ,  $\text{Tor}_n(M, N) = 0 \ \forall n \geq 2$  since  $M$  has a free res. of length 1 (Prop 8). So we write  $\text{Tor}(M, N) := \text{Tor}_1(M, N)$ .

**Lemma 11**  $f: M \rightarrow N$   $R$ -linear,  $P$   $R$ -module  $\Rightarrow$   
 $(\text{Coker } f) \otimes P \cong \text{Coker}(f \otimes \text{id}_P)$ . **Proof** Exercise.

**Proof of the UCT** (1) Constructing the SES

$$\underbrace{B_n = \text{im } d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \text{ker } d_n}_{n\text{-cycles}}$$

Make  $B_n, Z_n$  into chain complexes, taking 0 as differential.

There is a SES of chain complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_{n+1} & \xrightarrow{\text{incl}} & C_{n+1} & \xrightarrow{d} & B_n \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow d & & \downarrow 0 \\ 0 & \longrightarrow & Z_n & \xrightarrow{\text{incl}} & C_n & \xrightarrow{d} & B_{n-1} \longrightarrow 0 \\ & & & & \vdots & & \end{array}$$