Proof of the UCT (1) Constructing the SES

15 March

$$\underbrace{B_n = \operatorname{im} d_{n+1}}_{n\text{-boundaries}} \subseteq \underbrace{Z_n = \ker d_n}_{n\text{-cycles}}$$

Make B_n, Z_n into chain complexes, taking 0 as differential.

There is a SES of chain complexes:

$$\begin{tikzcd}
 & \vdots & & & \\
0 \arrow[r] & Z_{n+1} \arrow[r, "\text{incl}"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d"] \arrow[d, "d"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "\text{incl}"] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd}$$

B_n free by Thm 7 $\Rightarrow$ each row splits $\Rightarrow$ tensoring with M preserves exactness (Exercise). The SES $\otimes M$ induces a LES:

$$\cdots \to B_n \otimes M \xrightarrow{\tau} Z_n \otimes M \longrightarrow \frac{\ker d_n \otimes \mathrm{id}_M}{\operatorname{im} d_{n+1} \otimes \mathrm{id}_M} \longrightarrow B_{n-1} \otimes M \xrightarrow{\text{incl} \otimes \mathrm{id}_M} Z_{n-1} \otimes M \to \cdots$$

$$\Rightarrow \text{SES} \quad 0 \to \underbrace{H_n(C) \otimes M}_{\cong\, \operatorname{coker} \tau \text{ by Lemma 11}} \to \underset{\cong}{H_n(C;M)} \to \ker \text{incl} \otimes \mathrm{id}_M \to 0$$

There is a SES

$$0 \to B_{n-1} \xrightarrow{\text{incl}} Z_{n-1} \longrightarrow H_{n-1}(C) \to 0$$

which is a free resolution of $H_{n-1}(C)$. So

$$\ker \text{incl} \otimes \mathrm{id}_M \cong \operatorname{Tor}(H_{n-1}(C), M).$$

(1) The SES splits C_n free $\Rightarrow \exists\ p_n : C_n \to Z_n$ s.t. $\mathrm{incl} \circ p_n = \mathrm{id}_{Z_n}$. Correction 5 April $p : C \to Z$ is in general not a chain map! (Indeed, p chain map $\Rightarrow$ differential of C is zero). Proceed instead as follows: Let $\pi_n : Z_n \to H_n(C) = Z_n / B_n$ be the projection. Then $\pi_n \circ p_n$ is a map $C_n \to H_n(C)$, and this is a chain map when one considers $H_n(C)$ as complex with zero differential (since for $x \in C_n$: $d_n(x) \in B_{n-1} \subseteq Z_{n-1}$, so $p_{n-1}(d_n(x)) = d_n(x)$ and $\pi_{n-1}(p_{n-1}(d_n(x))) = [d_n(x)] = 0$). Thus $(\pi_n \circ p_n) \otimes \mathrm{id}_M : C_n \otimes M \to H_n(C) \otimes M$ is also a chain map, inducing a map $H_n(C; M) \xrightarrow{q} H_n(C) \otimes M$ on homology. To see that q is a splitting map, check that $q([x \otimes m]) = [x] \otimes m$ for all $x \in Z_n$ and $m \in M$.

(2) Naturality (Sketch)

$f : C \to C'$ chain map $\Rightarrow f(Z) \subseteq Z'$, $f(B) \subseteq B'$.

So f induces a map between the SES of chain complexes $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to Z'_n \to C'_n \to B'_{n-1} \to 0$, also after $\otimes M$, and so also between the associated LES, and so also between the SES in the UCT.

(3) Unnaturality of splitting: Exercise 2.4

Prop 12 $\mathrm{Tor}_0(M, N) \cong M \otimes N$

Proof $\cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$ deleted free res of M.

$\Rightarrow \mathrm{Tor}_0(M,N) = \mathrm{coker}(d_1 \otimes \mathrm{id}_N) \cong \mathrm{coker}(d_1) \otimes N$

$= H_0(F^M) \otimes N = M \otimes N$ $\square$

Remark 13 For $f : M \to M'$, $g : N \to N'$, one may set $\mathrm{Tor}_n(f,g) : \mathrm{Tor}_n(M,N) \to \mathrm{Tor}_n(M',N')$ to be given by $(\hat{f} \otimes g)_*$. Fixing one argument then makes Tor_n into an additive functor $R\text{-Mod} \to R\text{-Mod}$.

Prop 14 Let A, B, C be abelian groups.

(1) B free $\Rightarrow$ $\mathrm{Tor}(A,B) \cong 0$

(2) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then

$$0 \to \mathrm{Tor}(D,A) \to \mathrm{Tor}(D,B) \to \mathrm{Tor}(D,C) \to D\otimes A \to D \otimes B \to D \otimes C \to 0$$

is exact.

(3) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

(4) B torsion-free $\Rightarrow$ $T(A,B) \cong 0$

(5) $T(A,B) \cong \mathrm{Tor}(T(A), T(B))$.

(6) $\mathrm{Tor}(\mathbb{Z}/n, A) \cong \{x \in A \mid nx = 0\}$

(7) $\mathrm{Tor}(A \oplus B, C) \cong \mathrm{Tor}(A,C) \oplus \mathrm{Tor}(B,C)$

(8) $\mathrm{Tor}(A,B) \cong T(A) \otimes T(B)$ if A and B are f.g.

Proof (1) $0 \to F_1 \to F_0 \to A \to 0$ free res of A $\Rightarrow$
$0 \to F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0$ is exact $\Rightarrow \mathrm{Tor}(A,B) \cong 0$.

(2) Pick free res $0 \to F_1 \xrightarrow{d_1} F_0 \to D \to 0$

$$\Rightarrow \begin{array}{ccccccccc}
0 & \to & F_1 \otimes A & \xrightarrow{id_{F_1}\otimes f} & F_1 \otimes B & \xrightarrow{id_{F_1}\otimes g} & F_1 \otimes C & \to & 0 \\
 & & \downarrow{\scriptstyle d_1\otimes id_A} & & \downarrow{\scriptstyle d_1 \otimes id_B} & & \downarrow{\scriptstyle d_1\otimes id_C} & & \\
0 & \to & F_0 \otimes A & \xrightarrow[id_{F_0}\otimes f]{} & F_0 \otimes B & \xrightarrow[id_{F_0}\otimes g]{} & F_0 \otimes C & \to & 0
\end{array}$$

commutes and has exact rows. It is a SES of chain complexes! (Each complex made of two groups). The associated LES in homology is the desired sequence. ✓

(3) Apply (1) to a free res $0 \to F_1 \xrightarrow{d_1} F_0 \longrightarrow B \longrightarrow 0$

$\rightsquigarrow$ LES

$$0 \to \underbrace{\mathrm{Tor}(A,F_1)}_{0 \text{ because } F_1 \text{ free}} \to \underbrace{\mathrm{Tor}(A,F_0)}_{0 \text{ because } F_0 \text{ free}} \to \mathrm{Tor}(A,B) \to$$

$$\to A\otimes F_1 \xrightarrow{id_A \otimes d_1} A\otimes F_0 \longrightarrow A\otimes B \longrightarrow 0$$

$\Rightarrow$ $\mathrm{Tor}(A,B) \cong \ker(id_A \otimes d_1) = \mathrm{Tor}(B,A)$ by def of Tor,
using $A\otimes B \cong B \otimes A$. ✓

(4) Pick free res $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \longrightarrow 0$.

It's enough to show that $F_1 \otimes B \longrightarrow F_0 \otimes B$ is injective.

So let $\alpha \in F_1 \otimes B$ with $d_1 \otimes id_B(\alpha) = 0$ be given. To show: $\alpha = 0$.

<u>Claim</u> There is a f.g. subgroup $B' \subseteq B$ with $\alpha \in B'$ and $d_1 \otimes id_{B'}(\alpha) = 0$.

<u>Pf that Claim $\Rightarrow \alpha = 0$</u> B torsion free $\Rightarrow$ B' torsion free. B' torsion free and f.g.
$\Rightarrow$ B' free by classification of f.g. ab. groups. We already know that tensoring with a free module is exact $\Rightarrow$ $d_1 \otimes id_{B'}$ injective $\Rightarrow \alpha = 0$. ✓

<u>Pf of Claim</u> Use construction of $\otimes$: $F_0 \otimes B \cong$ free module $U_{F_0,B}$ with basis $F_0 \times B$ modulo submodule $I_{F_0,B} \subseteq U$ generated by

$$\left.\begin{array}{l} (\lambda x + x', y) - \lambda(x,y) - (x',y) \\ (x, \lambda y + y') - \lambda(x,y) - (x,y') \end{array}\right. \qquad (*)$$

Write $\alpha = \sum_{i=1}^{m} f_i \otimes b_i$. Then $d_1 \otimes id_B(\alpha) = 0 \Leftrightarrow \sum d_1(f_i)\otimes b_i = 0$

$$\Leftrightarrow \sum_{i=1}^{m} (d_1(f_i), b_i) = \sum_{j=1}^{k} \text{elements of the form } (*) \in I_{F_0,B}$$

Let $B' \subseteq B$ be generated by $b_1, \dots, b_m$ and all elements of B appearing in the sum on the RHS. Then $\alpha \in F_1 \otimes B'$, and
$d_1 \otimes id_{B'}(\alpha) = 0$ ✓

the following proofs were skipped in the lecture

(5) Apply (2) to the SES $0 \to T(B) \to B \to B/T(B) \to 0$:

$$0 \to \mathrm{Tor}(A, T(B)) \to \mathrm{Tor}(A,B) \to \underbrace{\mathrm{Tor}(A, B/T(B))}_{\substack{0 \text{ by (4) since} \\ B/T(B) \text{ torsion-free}}} \to \cdots$$

$\Rightarrow \mathrm{Tor}(A, T(B)) \cong \mathrm{Tor}(A,B)$. Now use (3) and repeat the argument. ✓

(6) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ is a free res of $\mathbb{Z}/n$.

$\Rightarrow \mathrm{Tor}(\mathbb{Z}/n, A) \cong \ker(A \xrightarrow{n} A) = \{x \in A \mid nx = 0\}$ ✓

(7) $\left.\begin{array}{l} 0 \to F_1 \to F_0 \to A \to 0 \\ 0 \to G_1 \to G_0 \to B \to 0 \end{array}\right\}$ free res.

$\Rightarrow 0 \to F_1 \oplus G_1 \to F_0 \oplus G_0 \to A \oplus B \to 0$ free res

Now $\mathrm{Tor}(A \oplus B, C) \cong \ker((F_1 \oplus G_1) \otimes C \to (F_0 \oplus G_0) \otimes C)$

$$\cong \ker(F_1 \otimes C \to F_0 \otimes C) \oplus \ker(G_1 \otimes C \to G_0 \otimes C)$$

$$\cong \mathrm{Tor}(A, C) \oplus \mathrm{Tor}(B, C)$$ ✓

(8) Using (7), (3), (1) and the classification of f.g. ab groups, it is enough to check this for $A \cong \mathbb{Z}/a$, $B \cong \mathbb{Z}/b$. This will be an Exercise on Sheet 3. ✓

□