5 Cohomology

Goal Dualize the singular chain complex, ie apply $\mathrm{Hom}(-,\mathbb{Z})$ (or $\mathrm{Hom}(-,M)$ for any abelian group M) $\rightsquigarrow$ cochain complex with cohomology. Why? Cohomology ...

* ... has more structure than a homology (it is a ring!)
* ... may arise in a natural way from geometric applications

Def A **cochain complex** C over a commutative ring R is a collection C^n of R-modules for $n \in \mathbb{Z}$ called **cochain modules**, R-linear maps $d^n : C^n \to C^{n+1}$ with $d^{n+1} \circ d^n = 0$ called **differentials**.

The **n-th cohomology module** of C is

$$H^n(C) = \frac{\overbrace{\ker d^n}^{n\text{-cocycles}}}{\underbrace{\operatorname{im} d^{n-1}}_{n\text{-coboundaries}}}$$

A **cochain map** $f : C \to D$ is a collection of R-linear $f^n : C^n \to D^n$ st $f^{n+1} \circ d^n_C = d^n_D \circ f^n \ \forall n$.

$f, g : C \to D$ are **homotopic**, written $f \simeq g$, if $\exists$ a **homotopy** $h : C \to D$, ie a collection of R-linear $h^n : C^n \to D^{n-1}$, s.t. $f_n - g_n = d^{n-1}_D \circ h_n + h_{n+1} \circ d^n_C$

Remark 1 C cochain complex

$\Longleftrightarrow$ D with $D_n = C^{-n}$, $d^D_n = d^{-n}_C$ is a chain complex

Under this 1:1-correspondence, cohomology $\leftrightarrow$ homology, cochain maps $\leftrightarrow$ chain maps, homotopies $\leftrightarrow$ homotopies etc.

So everything that is true for chain complexes also holds true mutatis mutandis for cochain complexes, eg Prop 2.

Prop 2 (1) $f: C \to D$ a cochain map $\Rightarrow$

$f^*: H^n(C) \to H^n(D)$, $f^*([x]) = [f(x)]$ is a well-def. R-homom.

(2) $H^n(-)$ is an additive functor

$$\underbrace{CoCh(R)}_{\text{category of cochain complexes over } R,\text{ cochain maps}} \longrightarrow R\text{-Mod}$$

(3) $f \simeq g \Rightarrow f^* \simeq g^*$.

No proof

Prop 3 If $F: R\text{-Mod} \to R\text{-Mod}$ is a <u>contravariant</u> additive functor, then $F: Ch(R) \longrightarrow CoCh(R)$ is also contravariant additive:

$$\ldots C_n \xrightarrow{d_n} C_{n-1} \ldots \longmapsto \ldots F(C_n) \xleftarrow{F(d_n)} F(C_{n-1}) \ldots$$

cochain complex $F(C)$
with $F(C)^n = F(C_n)$,
$d^n_{F(C)} = F(d^{n-1}_C)$

No proof

Def X top. space, $A \subseteq X$, M an abelian group.
Then the cochain complex obtained from $C_n(X,A)$ by applying $\mathrm{Hom}(-, M)$ is called the singular cochain complex of (X,A) with coefficients in M, denoted $C^n(X,A;M)$ and its cohomology the singular cohomology of (X,A) with coefficients in M, denoted $H^n(X,A;M)$. We may drop "$;M$" for $M = \mathbb{Z}$.
For $f: (X,A) \to (Y,B)$ continuous, write f^c for the cochain map $C^n(Y,B;M) \to C^n(X,A;M)$,
$f^c = \mathrm{Hom}(f_c, M)$, and f^* for the induced homom. $H^n(Y,B;M) \longrightarrow H^n(X,A;M)$.

Ex 4 $C^0(X;M) = \mathrm{Hom}(C_0(X), M)$. Corresponds to functions $X \to M$. Let $\varphi \in C^0(X;M)$. Then $d^0(\varphi)$ sends $\sigma: \Delta^1 = [0,1] \to M$ to $\varphi(d_1(\sigma)) = \varphi(\sigma(1)) - \varphi(\sigma(0))$.

So $d^0(\varphi) = 0 \iff \varphi(\sigma(0)) = \varphi(\sigma(1))\ \forall \sigma \iff \varphi$ constant on path-connected components. Hence

$$H^0(X;M) = \ker d^0 \cong \prod_{\pi_0(X)} M$$

note: for $\pi_0(X)$ infinite $H^0(X;\mathbb{Z}) \not\cong H_0(X;\mathbb{Z})$, where $H^0(X;\mathbb{Z}) \cong \prod_{\pi_0(X)} \mathbb{Z}$ and $H_0(X;\mathbb{Z}) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$

Rmk 5 A hands-on approach to cochains:

An n-cochain $\varphi \in C^n(X;M)$ is a homom. $C_n(X) \to M$.

So n-chains correspond to functions

$$\{\text{singular } n\text{-simplices } \sigma: \Delta^n \to X\} \longrightarrow \mathbb{Z}$$

The $(n+1)$-cochain $d^n(\varphi)$ sends $\tau: \Delta^{n+1} \to X$ to $\varphi(d_{n+1}(\tau))$.

So φ is an n-cocycle $\iff \varphi$ is zero on n-boundaries $\in B_n$.

φ is an n-coboundary $\Rightarrow \varphi(\sigma)$ is determined by $d_n(\sigma)$.

$\Longrightarrow \varphi$ is zero on n-cycles $\in Z_n$

Correction 22 April The implication "$\Leftarrow$" does not generally hold: there may be cochains φ that are zero on n-cycles, but that are not coboundaries. Indeed, this happens if φ is a cocycle, $[\varphi] \neq 0 \in H^n(X;M)$, and $ev([\varphi]) = 0$.

Thus: An n-cocycle φ induces a homom. $C_n(X)/B_n \to M$, by restriction it also induces a homom.

$$Z_n/B_n = H_n(X) \longrightarrow M.$$

For n-coboundaries φ, this homom. is zero. Thus we have a homom. called the evaluation homomorphism

$$ev: H^n(X;M) \longrightarrow \mathrm{Hom}(H_n(X), M)$$

which may be seen to be natural in both X and M.

Universal Coefficient Thm for Cohomology

Let C be a chain complex of free abelian groups and A an abelian group

(1) There is a split SES

$$0 \to \mathrm{Ext}(H_{n-1}(C), A) \to H^n(C; M) \xrightarrow[ev]{} \mathrm{Hom}(H_n(C), A) \to 0$$

to be defined!

(2) These SES are natural in C and A.

(3) The splittings cannot be chosen naturally