Def Let M, N be R-modules, and $\mathcal{F}$ a free res. of M. Then let

$$\operatorname{Ext}^n_R(M,N) := H^n(\operatorname{Hom}(\mathcal{F}^M, N))$$

$\mathcal{F}^M$ unique up to hom. equiv. $\Longrightarrow$ Def of Ext independent of choice of $\mathcal{F}$.

As with Tor, we have:

* $\operatorname{Ext}^0_R(M,N) \cong \operatorname{Hom}(M,N)$.
* $\operatorname{Ext}^n_{\mathbb{Z}}(A,B) = 0$ for all $n \geq 2$, so we write $\operatorname{Ext}(A,B)$ for $\operatorname{Ext}^1_{\mathbb{Z}}(A,B)$.

For the proof of the first point, one needs:

Lemma 6 M, N, P R-modules, $f: M \to N$ R-linear

$$\Rightarrow \operatorname{Hom}(\operatorname{coker} f, P) \cong \ker(\operatorname{Hom}(f, P))$$

Proof $M \longrightarrow N \longrightarrow \operatorname{coker} f \longrightarrow 0$ exact

$\Rightarrow 0 \to \operatorname{Hom}(\operatorname{coker} f, P) \longrightarrow \operatorname{Hom}(N,P) \to \operatorname{Hom}(M,P)$ is exact

(same argument as in Ex Sheet 1, 2b) □

Rmk 7 * Ext is not symmetric: $\operatorname{Ext}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$

$\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/n) \cong 0$

(as we shall see from Prop 8)

* Ext can behave unexpectedly:

$$\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \text{uncountably-dimensional } \mathbb{Q}\text{-vector space}$$

Prop 8 For all ab groups A, B, C, the following hold:

(1) $\mathrm{Ext}(A \oplus B, C) \cong \mathrm{Ext}(A, C) \oplus \mathrm{Ext}(B, C)$

(2) $\mathrm{Ext}(A, B \oplus C) \cong \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A, C)$

(3) A free $\Rightarrow \mathrm{Ext}(A, B) \cong 0$.

(4) $\mathrm{Ext}(\mathbb{Z}/n, A) \cong A/nA$

Note this suffices to compute $\mathrm{Ext}(\text{f.g. group}, A)$.

(5) $\mathrm{Ext}(A, B) \cong T(A) \otimes B$ if A, B f.g.

Compare (4), (5) to Tor: $\mathrm{Tor}(\mathbb{Z}/n, A) \cong \{x \in A \mid nx = 0\}$

$\mathrm{Tor}(A, B) \cong T(A) \otimes T(B)$ for A, B f.g.

Proof of (4) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ free res. $\mathcal{F}$

$$\mathrm{Hom}(\mathcal{F}^{\mathbb{Z}/n}, A) = 0 \leftarrow \underset{\cong A}{\mathrm{Hom}(\mathbb{Z}, A)} \xleftarrow{n} \underset{\cong A}{\mathrm{Hom}(\mathbb{Z}, A)} \leftarrow 0$$

$\Rightarrow \mathrm{Ext} = H^1$ of this cochain complex $\cong A/nA$ $\square$

Rmk 9 Let R-modules M, N be given. An extension of N by M is a SES $0 \to N \to P \to M \to 0$. It is equivalent to another extension $0 \to N \to P' \to M \to 0$ if $\exists f: P \to P'$ st

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "id_N"'] & P \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "id_M"] & 0 \\ 0 \arrow[r] & N \arrow[r] & P' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

commutes, Five-Lemma $\Rightarrow f$ is iso. So equivalence is an equiv. rel.

One finds $\{\text{Extensions of } N \text{ by } M\}/\text{equiv} \xleftrightarrow{1:1} \mathrm{Ext}^1_R(M, N)$.

Prop 10 Assume $H_n(X,A)$ is f.g. for all n. Then
$$H^n(X,A;\mathbb{Z}) \cong \underbrace{F(H_n(X,A))}_{\text{free part}} \oplus T(H_{n-1}(X,A))$$
$F(B) := B/T(B)$

Proof UCT $\Rightarrow$ $H^n(X,A;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X,A),\mathbb{Z})$
$\oplus \mathrm{Ext}(H_{n-1}(X,A),\mathbb{Z})$

$\cong \mathrm{Hom}(F(H_n(X,A)),\mathbb{Z})$ $\cong F(H_n(X,A))$
$\oplus \mathrm{Hom}(T(H_n(X,A)),\mathbb{Z})$ $\cong 0$
$\oplus \mathrm{Ext}(F(H_{n-1}(X,A)),\mathbb{Z})$ $\cong 0$
$\oplus \mathrm{Ext}(T(H_{n-1}(X,A)),\mathbb{Z})$ $\cong T(H_{n-1}(X,A))$ □

Def The cellular cochain complex $C^\bullet_{cw}(X)$ of a CW-complex X is $\mathrm{Hom}(C^{cw}_\bullet(X), M)$. Its cohomology $H^n_{cw}(X;M)$ is the n-th cellular cohomology group.

Thm 11 $H^n_{cw}(X;M) \cong H^n(X;M)$.

Example 12 $C^{cw}_\bullet(\mathbb{RP}^2) = 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

$H_0^{cw} \cong \mathbb{Z}$, $H_1^{cw} \cong \mathbb{Z}/2$, $H_2^{cw} = 0$

Hands-on Trick: C a chain complex of f.g. free ab. groups with a chosen basis, then

$$\underset{\text{wrt to the basis}}{(\text{Matrix of } d_n)^T} = \underset{\text{wrt the dual basis}}{\text{Matrix of } \mathrm{Hom}(d_n,\mathbb{Z})}$$

$\Rightarrow C^\bullet_{cw}(\mathbb{RP}^2;\mathbb{Z}) = 0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$

and $H^0_{cw} \cong \mathbb{Z}$, $H^1_{cw} \cong 0$, $H^2_{cw} \cong \mathbb{Z}/2$