Proof of UCT (1)

There is a SES of chain complexes:

$$\begin{tikzcd} & \vdots & & & \\ 0 \arrow[r] & Z_{n+1} \arrow[r, "\mathrm{incl}"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_{n+1}"'] & B_n \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "\mathrm{incl}"] & C_n \arrow[r, "d_n"] & B_{n-1} \arrow[r] & 0 \\ & & \vdots & & \end{tikzcd}$$

B_n free by Thm 4.7 $\Rightarrow$ each row splits $\Rightarrow$ SES of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(B_{n-1},M) \arrow[r, "d^{n-1}"] \arrow[d, "0"'] & \mathrm{Hom}(C_n,M) \arrow[r, "\mathrm{incl}^*"] \arrow[d, "d^n"'] & \mathrm{Hom}(Z_n,M) \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(B_n,M) \arrow[r, "d^n"] & \mathrm{Hom}(C_{n+1},M) \arrow[r, "\mathrm{incl}^*"] & \mathrm{Hom}(Z_{n+1},M) \arrow[r] & 0 \end{tikzcd}$$

This induces a LES

$$\ldots \to \mathrm{Hom}(Z_{n-1}, M)$$
$$\xrightarrow{\partial^{n-1}} \mathrm{Hom}(B_{n-1}, M) \to H^n(C;M) \to \mathrm{Hom}(Z_n,M)$$
$$\xrightarrow{\partial^n} \mathrm{Hom}(B_n, M) \to \ldots$$

Check that $\partial^i = \mathrm{Hom}(B_n \hookrightarrow Z_n, M)$

$\Rightarrow$ SES

$$0 \to \underbrace{\mathrm{coker}\, \partial^{n-1}}_{\cong \mathrm{Ext}(H_{n-1}(C),M)} \to H^n(C;M) \to \underbrace{\ker \partial^n}_{\cong \mathrm{Hom}(\mathrm{coker}\, B_n \to Z_n, M)} \to 0$$

$\cong \mathrm{Hom}(\mathrm{coker}\, B_n \to Z_n, M)$ (by Lemma 6)

$\cong \mathrm{Hom}(H_n(C), M)$

because: free res $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$

$\rightsquigarrow$ cochain complex $0 \leftarrow \mathrm{Hom}(B_{n-1}, M) \xleftarrow{\partial^{n-1}} \mathrm{Hom}(Z_{n-1}, M)$

with $H^1 \cong \mathrm{coker}\, \partial^{n-1}$, and $H^1 \cong \mathrm{Ext}$ by def of Ext. $\square$

Prop 11 Singular cohomology satisfies axioms that are analogue to the Eilenberg-Steenrod axioms for homology (see ②):

Data

$H^n(-;M)$ are contravariant functors $\{\text{Pairs of Spaces}\} \to R\text{-Mod}$

There are natural connecting homom. $\partial: H^n(A;M) \longrightarrow H^{n+1}(X,A;M)$

Axioms

Homotopy (1) $f \simeq g \Rightarrow f^* = g^*$

Excision (2) $\bar{U} \subseteq A^\circ \Rightarrow \text{incl}^*: H^n(X,A;M) \longrightarrow H^n(X \setminus U, A \setminus U; M)$ iso

Dimension (3) $H^n(\{*\};M) \cong M$ for $n=0$, trivial for $n \neq 0$.

Additivity (4) $H^n(\coprod_\alpha X_\alpha; M) \xrightarrow{i} \prod_\alpha H^n(X_\alpha; M)$ is an iso,

with i given by $i_\alpha = (\text{inclusion } X_\alpha \longrightarrow \coprod_\alpha X_\alpha)^*$.

Exactness (5) There are LESs

$$\dots \to H^n(X,A;M) \xrightarrow{\text{incl}^*} H^n(X;M) \xrightarrow{\text{incl}^*} H^n(A;M) \xrightarrow{\partial} H^{n+1}(X,A;M) \to \dots$$

Similarly as for Homology with coefficients, all axioms follow more or less directly from homotopy equivalences of singular chain complexes being send to hom. equiv. of singular cochain complexes by the additive $\text{Hom}(-,M)$ functor.

Proof of (4) Alg Top I: $\sum_\alpha (\mathrm{incl}_\alpha)_c : \bigoplus_\alpha C_\bullet(X_\alpha) \longrightarrow C_\bullet(\coprod_\alpha X_\alpha)$

is a homotopy equivalence $\Rightarrow$ so is

$$\begin{tikzcd}
\mathrm{Hom}(C_\bullet(\coprod_\alpha X_\alpha), M) \arrow[r, "\mathrm{Hom}(\sum_\alpha (\mathrm{incl}_\alpha)_c, M)"] \arrow[dd, "\text{iso (by def)}"'] & \mathrm{Hom}(\bigoplus_\alpha C_\bullet(X_\alpha), M) \arrow[d, "\text{iso}"] \\
& \prod_\alpha \mathrm{Hom}(C_\bullet(X_\alpha), M) \arrow[d, "\text{iso (by def)}"] \\
C^\bullet(\coprod_\alpha X_\alpha; M) \arrow[r, "\alpha\text{-component is } (\mathrm{incl}_\alpha)_c"'] & \prod_\alpha C^\bullet(X_\alpha; M)
\end{tikzcd}$$

$\square$

Further good properties of cohomology:

Thm 12 (Mayer-Vietoris) $A, B \subseteq X$, $A^\circ \cup B^\circ = X \Rightarrow$ LES

$$\dots \to H^n(X;M) \longrightarrow H^n(A;M) \oplus H^n(B;M) \longrightarrow H^n(A\cap B;M) \longrightarrow H^{n+1}(X) \to \dots$$

Remark 13 Understanding the connecting homomorphisms in the Mayer-Vietoris-sequence:

<u>Homology</u> $H_n(X) \longrightarrow H_{n-1}(A\cap B)$:

Represent a homology class $[x] \in H_n(X)$ as $[y+z]$, where $y \in C_n(A)$ and $z \in C_n(B)$. (Here, we abuse notation and write y also for the image of y under $C_n(A) \hookrightarrow C_n(X)$, similarly for z.) Now send $[x] \mapsto [dy]$.

(since $0 = dx = d(y+z) \Rightarrow dy = -dz$, so $dy \in C_{n-1}(A\cap B)$, again abusing notation).

See Hatcher p. 150

A similar understanding for cohomology is more complicated. The following wasn't discussed in the lecture.

Cohomology $H^n(A \cap B) \longrightarrow H^{n+1}(X)$:

Extend a cohomology class $[\varphi] \in H^n(A \cap B)$, which is a map $C_n(A \cap B) \to \mathbb{Z}$, to a map $\psi : C_n(A) \to \mathbb{Z}$, ie a cochain $\psi \in C^n(A)$.

Correction 30 April

For each $x \in C_{n+1}(X)$, choose $y \in C_{n+1}(A)$, $z \in C_{n+1}(B)$ such that $x - (y+z)$ is a boundary. Then send $[\varphi]$ to the cohomology class in $H^{n+1}(X)$ that sends each x to $\psi(\partial y)$.

Thm 14 (Good Pairs) $A \subseteq X$ non-empty closed, A a deformation retract of an open neighbourhood of A in X $\Rightarrow$ the projection $(X,A) \to (X/A, \{*\})$ induces an iso

$$\underbrace{H^n(X/A, \{*\})}_{\cong \tilde{H}^n(X/A)} \longrightarrow H^n(X, A)$$

Def For $X \neq \emptyset$, the n-th reduced cohomology group $\tilde{H}^n(X;M)$ is the n-th cohomology group of the augmented cochain complex

$$0 \to M \xrightarrow{\varepsilon} C^0(X;M) \longrightarrow C^1(X;M) \longrightarrow \dots$$

with $\varepsilon(m)(\sigma) = m$ for all $\sigma : \Delta^0 \to X$.

Prop 15 $H^n(X;M) \cong \tilde{H}^n(X;M)$ for $n \geq 1$,

$H^0(X;M) \cong \tilde{H}^0(X;M) \oplus M$

Ex 16 $\tilde{H}^n(S^k) \cong \mathbb{Z}^{\delta(n,k)}$

$k=0$: ✓. Assume now $k \geq 1$.

1st Proof $C^\bullet_{cw}(S^k) \cong \mathrm{Hom}(C^{cw}_\bullet(S^k), \mathbb{Z}) \cong C^{cw}_\bullet(S^k)$

2nd Proof $H_\bullet(S^k)$ free $\xRightarrow{UCT}$ $H^n(S^k) \cong H_n(S^k)$

3rd Proof $A = S^k \setminus \{e_1\}$, $B = S^k \setminus \{-e_1\}$ $\Rightarrow$ A, B contractible

$\Rightarrow$ Mayer-Vietoris gives iso $H^i(\underbrace{A \cap B}_{\simeq S^{k-1}}) \longrightarrow H^{i+1}(S^k)$

Proceed by induction.

4th Proof

$$\begin{tikzcd}
H^i(S^k) \arrow[d, "\text{iso}"', "\text{LES of Pair } (D^{k+1}, S^k)"] \\
H^{i+1}(D^{k+1}, S^k) \arrow[d, "\text{iso}"', "\text{due to good pair}"] \\
H^{i+1}(S^{k+1})
\end{tikzcd}$$

Prop 17 Let $n \geq 1$. If $f: S^n \to S^n$ has degree $k \in \mathbb{Z}$, then $f^*: H^n(S^n) \longrightarrow H^n(S^n)$ is multiplication by k.

Reminder "f has degree k" is by def equivalent to:

$f_*: H_n(S^n) \to H_n(S^n)$ is multiplication by k

1st Proof

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & C^{cw}_n(S^n) \arrow[r, "0"] \arrow[d, "f_c = \text{mult by } k"] & \cdots \\
\cdots \arrow[r, "0"] & C^{cw}_n(S^n) \arrow[r, "0"] & \cdots
\end{tikzcd}
\xrightarrow[\mathrm{Hom}(\cdot, \mathbb{Z})]{\text{apply functor}}
\begin{tikzcd}
\cdots & C^n_{cw}(S^n) \arrow[l, "0"'] \arrow[d, "f^c = \mathrm{Hom}(f_c, \mathbb{Z}) = \text{mult by } m"] & \cdots \arrow[l, "0"'] \\
\cdots & C^n_{cw}(S^n) \arrow[l, "0"'] & \cdots \arrow[l, "0"']
\end{tikzcd}$$

2nd Proof Use naturality of UCT. (Skipped in lecture)

$\mathrm{Ext}(H_{n-1}(S^n), \mathbb{Z}) \cong 0$ since $H_{n-1}(S^n)$ is free (namely, it is 0 (if $n \geq 2$) or $\mathbb{Z}$ (if $n=1$)). So we have an iso

$$ev : H^n(S^n) \longrightarrow \mathrm{Hom}(H_n(S^n), \mathbb{Z})$$

It is natural, so the following commutes:

$$\begin{tikzcd} H^n(S^n) \arrow[r, "ev"', "iso"] \arrow[d, "f^*"] & \mathrm{Hom}(H_n(S^n), \mathbb{Z}) \arrow[d, "{\mathrm{Hom}(f_*, \mathbb{Z}) = \text{mult by } k}"] \\ H^n(S^n) \arrow[r, "ev", "iso"'] & \mathrm{Hom}(H_n(S^n), \mathbb{Z}) \end{tikzcd}$$

$\square$