(6) The cup product

Reminder about simplexes If $v_0, \dots, v_n \in \mathbb{R}^\ell$ s.t. $v_1 - v_0, \dots, v_n - v_0$ are lin indep., then the convex hull of $\{v_0, \dots, v_n\}$, ie

$$\left\{ \sum_{i=0}^{n} \lambda_i v_i \;\middle|\; \sum_{i=0}^{n} \lambda_i = 1 \,,\; (\lambda_0, \dots, \lambda_n) \in [0,1]^{n+1} \right\} \subseteq \mathbb{R}^\ell$$

together with the tuple $(v_0, \dots, v_n)$, is called an **n-simplex**, denoted **$[v_0, \dots, v_n]$**. Every pair of n-simplexes $[v_0, \dots, v_n]$, $[v_0', \dots, v_n']$ is naturally homeomorphic via $\sum \lambda_i v_i \longmapsto \sum \lambda_i v_i'$.

The **standard n-simplex** is $\Delta^n := [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$.

A **singular n-simplex** of a top. space X is a cont. map $\sigma : \Delta^n \to X$.

They form the basis of $C_n(X)$. The **boundary operator** $d : C_n(X) \to C_{n-1}(X)$ is given by $d(\sigma) = \sum_{i=0}^{n} \sigma|_{\underbrace{[e_0, \dots, \hat{e}_i, \dots, e_n]}_{\text{means } e_i \text{ is left out}}}$.

(where we implicitly identify the non-standard simplex $[e_0, \dots, \hat{e}_i, \dots, e_n]$ with Δ^{n-1} via the natural homeo).

Throughout, let R be a commutative unital ring.

Def X top space, $\varphi \in C^n(X;R)$, $\psi \in C^k(X;R)$.

Let the **cup-product $\varphi \smile \psi$** $\in C^{n+k}(X;R)$

\smile, not \cup, in LaTeX

be given sending singular simplexes $\sigma : \Delta^{n+k} = [e_0, \dots, e_{n+k}] \to X$ to

$$(\varphi \smile \psi)(\sigma) = \varphi\Big(\underbrace{\sigma|_{[e_0, \dots, e_n]}}_{\text{"front face" of } \sigma}\Big) \underbrace{\cdot}_{\text{multiplication in } R} \psi\Big(\underbrace{\sigma|_{[e_n, \dots, e_{n+k}]}}_{\text{"back face" of } \sigma}\Big)$$

Prop 1 (1) $\smile : C^n(X;R) \times C^k(X;R) \longrightarrow C^{n+k}(X;R)$

is R-bilinear. (uses distributivity & associativity of R)

(2) $\smile$ is associative: $(\varphi \smile \psi) \smile \eta = \varphi \smile (\psi \smile \eta)$

(uses associativity of R)

(3) Let $\varepsilon \in C^0(X;R)$, $\varepsilon(\sigma) = 1 \in R$ for all σ. Then

$\varphi \smile \varepsilon = \varepsilon \smile \varphi = \varphi$. (uses unit of R)

Proof Exercise.

Remark 2 $\smile$ makes $C^\bullet(X;R) = \bigoplus_{n=0}^{\infty} C^n(X;R)$ into a

(generally non-commutative) unital R-algebra (by Prop 1).

Moreover, $C^\bullet(X;R)$ is graded:

a **grading** on an R-algebra S is a decomposition

$S = \bigoplus_{n \in \mathbb{Z}} S_n$ as an R-module, such that $S_n S_k \subseteq S_{n+k}$.

We write **deg** $x = n$ for $x \in S_n$, $x \neq 0$. deg is not defined if $x \notin S_n$ $\forall n$.

Example 3 $C^\bullet(\emptyset; R)$ = the zero ring

$C^\bullet(\{*\}; R)$: For all $n \geq 0$, $C_n(\{*\})$ is generated by the

constant $\sigma_n : \Delta^n \to \{*\}$, and $C^n(\{*\};R)$ by $\varphi_n : \sigma_n \longmapsto 1$.

Check $\varphi_n \smile \varphi_k = \varphi_{n+k}$. So we have an isomorphism of graded

R-algebras: $C^\bullet(\{*\};R) \longrightarrow R[x]$, $\varphi_n \longmapsto x^n$.

Here, deg on $R[x]$ is different from the usual deg of polynomials:

$\deg(r x^n) = n$, deg not defined for non-monomials.

Prop 4 (Graded Leibniz rule). For $\varphi \in C^n(X;R)$, $\psi \in C^k(X;R)$:

$$d(\varphi \smile \psi) = (d\varphi) \smile \psi + (-1)^n \varphi \smile d\psi$$

Koszul sign rule:

"when d jumps over something of degree k, $(-1)^k$ appears"

Proof Calculate:

$$((d\varphi)\smile\psi)(\sigma:[e_0,\dots,e_{n+k+1}]\to X)$$

$$= (d\varphi)(\sigma|_{[e_0,\dots,e_{n+1}]})\cdot\psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

$$= \varphi(d\sigma|_{\dots})\cdot\psi(\dots)$$

$$= \varphi\Big(\sum_{i=0}^{n+1}(-1)^i\,\sigma|_{[e_0,\dots,\hat{e}_i,\dots e_{n+1}]}\Big)\cdot\psi(\dots)$$

$$= \sum_{i=0}^{n+1}(-1)^i\,\varphi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots e_{n+1}]})\,\psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

and:

$$(\varphi\smile d\psi)(\sigma) =$$

$$= \sum_{j=0}^{k+1}(-1)^j\,\varphi(\sigma|_{[e_0,\dots,e_n]})\,\psi(\sigma|_{[e_n,\dots,\widehat{e_{n+j}},\dots,e_{n+k+1}]})$$

Now plug this into:

$$((d\varphi)\smile\psi)(\sigma)+(-1)^n(\varphi\smile d\psi)(\sigma)$$

Notice the last summand $(i=n+1)$ cancels the first $(j=0)$!

$$= \sum_{i=0}^{n}(-1)^i\,\varphi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{n+1}]})\,\psi(\sigma|_{[e_{n+1},\dots,e_{n+k+1}]})$$

$$+ \sum_{m=n+1}^{n+k+1}(-1)^m\,\varphi(\sigma|_{[e_0,\dots,e_n]})\,\psi(\sigma|_{[e_n,\dots,\hat{e}_m,\dots,e_{n+k+1}]})$$

index shift $m = j+n$

$$= (d(\varphi\smile\psi))(\sigma)$$

$\square$

Prop 5 (1) cocycle $\smile$ cocycle = cocycle

(2) coboundary $\smile$ cocycle = coboundary and

cocycle $\smile$ coboundary = —— " ——

(3) For $[\varphi] \in H^n(X;R)$, $[\psi] \in H^k(X;R)$,

$[\varphi] \smile [\psi] := [\varphi \smile \psi] \in H^{n+k}(X;R)$ is well-def

(4) $\smile$ makes $H^\bullet(X;R) := \bigoplus_{i=0}^{\infty} H^i(X;R)$ into a graded R-algebra.

Proof (1) If $d\varphi = d\psi = 0 \Rightarrow d(\varphi \smile \psi) = (d\varphi) \smile \psi \pm \varphi \smile d\psi = 0$.

(2) If $\varphi = d\eta$ and $d\psi = 0 \Rightarrow \varphi \smile \psi = (d\eta) \smile \psi = d(\eta \smile \psi)$.

(3) $\varphi \smile \psi$ is a cocycle by (1).

If $\varphi' = \varphi + d\eta$, $\psi' = \psi + d\xi$, then

$$[\varphi' \smile \psi'] = [\varphi \smile \psi] + \underbrace{[\varphi \smile d\xi]}_{=0} + \underbrace{[d\eta \smile \psi']}_{=0} \quad \text{by (2)}$$

(4) Follows from Prop 1 $\square$