Example 6 If $\ell \geq 1$, then $H^\bullet(S^\ell; R) \cong R[x]/(x^2)$ with $\deg x = \ell$ ($x^2 = 0$ since ~~since~~ there is no non-trivial cohomology class of deg 2ℓ).

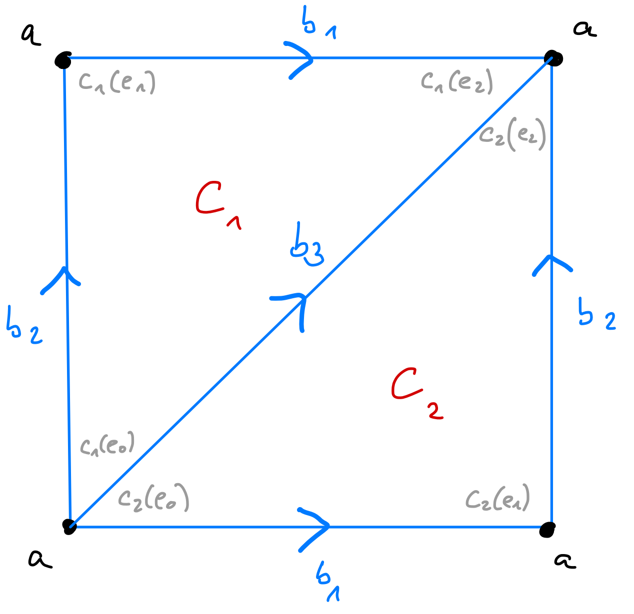
Def For a Δ-complex X, define $\smile$ in the same way as before on the simplicial cochain complex $C^\bullet_\Delta(X; R) = \mathrm{Hom}(C^\Delta_\bullet(X), R)$, and on its cohomology $H^\bullet_\Delta(X; R)$.

Prop 7 The chain homotopy equivalence **Thm 2.27 in Hatcher**
$C^\Delta_\bullet(X) \longrightarrow C_\bullet(X)$, sending simplex to simplex, induces a chain homotopy equivalence $C^\bullet(X) \longrightarrow C^\bullet_\Delta(X)$ that preserves the cup product.

Proof Immediate from def $\square$

Example 8 $X = S^1 \times S^1$. Know $H^0(X) \cong \mathbb{Z}$, $H^1(X) \cong \mathbb{Z}^2$, $H^2(X) \cong \mathbb{Z}$. So $\smile$ may be interesting on $H^1(X)$.

Put a Δ-complex-structure on X:



$a \in C^\Delta_0(X)$, $b_1, b_2, b_3 \in C^\Delta_1(X)$
$c_1, c_2 \in C^\Delta_2(X) \implies$

$db_i = 0$,
$dc_1 = dc_2 = b_1 - b_3 + b_2$

One computes that:

$H^\Delta_0(X; \mathbb{Z})$ has basis $[a]$
$H^\Delta_1(X; \mathbb{Z})$ — " — $[b_1], [b_2]$
$H^\Delta_2(X; \mathbb{Z})$ — " — $[c_1 - c_2]$

Since $H_\bullet^\Delta(X;\mathbb{Z})$ is torsion-free, the UCT implies $H^\bullet_\Delta(X;\mathbb{Z}) \cong \operatorname{Hom}(H_\bullet^\Delta(X;\mathbb{Z}))$. So the dual basis of the basis $[a], [b_1], [b_2], [c_1-c_2]$ is a basis for $H^\bullet_\Delta(X;\mathbb{Z})$:

$$\begin{array}{rcccc} & [\varphi], & [\psi^1], & [\psi^2], & [\eta] \\ \text{deg} & 0 & 1 & 1 & 2 \end{array}$$

with $\varphi(a) = 1$, $\psi^i(b_j) = \delta_{ij}$, $\eta(c_1 - c_2) = 1$.

Let's calculate $[\psi^1] \smile [\psi^2]$! Since $[\psi^1] \smile [\psi^2] \in H^2(X;\mathbb{Z})$

$\Rightarrow [\psi^1] \smile [\psi^2] = \lambda [\eta]$ for some $\lambda \in \mathbb{Z}$.

Evaluate both sides on $[c_1 - c_2]$:

$$\begin{aligned}
\lambda &= ev([\psi^1] \smile [\psi^2])([c_1 - c_2]) \\
&= ev([\psi^1 \smile \psi^2])([c_1 - c_2]) && \text{by def of } \smile \text{ on cohomology} \\
&= (\psi^1 \smile \psi^2)(c_1 - c_2) && \text{by def of ev} \\
&= (\psi^1 \smile \psi^2)(c_1) - (\psi^1 \smile \psi^2)(c_2) && \text{by linearity} \\
&= \psi^1(c_1|_{[e_0,e_1]})\, \psi^2(c_1|_{[e_1,e_2]}) - \psi^1(c_2|_{[e_0,e_1]})\, \psi^2(c_2|_{[e_1,e_2]}) && \text{by def of } \smile \text{ on cochains} \\
&= \psi^1(b_2)\psi^2(b_1) - \psi^1(b_1)\psi^2(b_2) \\
&= -1
\end{aligned}$$

$\Rightarrow [\psi^1] \smile [\psi^2] = -[\eta]$.

Similarly, one computes $[\psi^2] \smile [\psi^1] = [\eta]$

and $[\psi^i] \smile [\psi^i] = 0$.

So $H^\bullet(S^1 \times S^1; \mathbb{Z}) \cong \underbrace{\mathbb{Z}\langle x, y\rangle}_{\text{free algebra generated by } x,y.} / (xy = -yx,\ x^2 = y^2 = 0)$

Prop 9 (Naturality of $\smile$)

$f: X \longrightarrow Y$ cont. map of top. spaces, $[\varphi]\in H^n(Y;R), [\psi]\in H^k(Y;R)$

$$\Rightarrow f^*([\varphi]\smile[\psi]) = (f^*[\varphi])\smile(f^*[\psi])$$

Proof (skipped in the lecture)

For all $(n+k)$-simplexes σ: $(f^c(\varphi\smile\psi))(\sigma) = \varphi\smile\psi(f\circ\sigma)$

$$= \varphi(f\circ\sigma|_{[e_0,\dots,e_n]})\,\psi(f\circ\sigma|_{[e_n,\dots,e_{n+k}]})$$

$$= f^c\varphi(\sigma|_{\dots})\cdot f^c\psi(\sigma|_{\dots}) = ((f^c\varphi)\smile(f^c\psi))(\sigma).$$

Now $f^*([\varphi]\smile[\psi]) = f^*([\varphi\smile\psi]) = [f^c(\varphi\smile\psi)]$
$= [(f^c\varphi)\smile(f^c\psi)] = [f^c\varphi]\smile[f^c\psi] =$
$f^*([\varphi])\smile f^*([\psi])$ $\square$

In other words: f^* is a homomorphism of graded R-algebras!

Prop 10 X, Y top spaces $\Rightarrow$ We have graded R-algebra isos

(1) $H^\bullet(X\sqcup Y;R) \xrightarrow[\binom{incl^*}{incl^*}]{} H^\bullet(X;R)\times H^\bullet(Y;R)$

(2) $H^\bullet(X\vee Y;R) \xrightarrow[\binom{incl^*}{incl^*}]{}$ Subalgebra of $H^\bullet(X;R)\times H^\bullet(Y;R)$ containing in deg 0 only those (φ,ψ) with $\varphi(x_0)=\psi(y_0)$

wedge product $X\sqcup Y/_{\{x_0\}\sim\{y_0\}}$ for some $x_0\in X$, $y_0\in Y$ that are deformation retracts of neighbourhoods N_x, N_y.

Proof (1) We know $\binom{incl^*}{incl^*}$ is an R-module isom. (eg use MV).

It's an algebra homom by Prop 9.

(2) Mayer–Vietoris gives isos for $n\geq 1$, and a SES

$$0\to H^0(X\vee Y;R)\to H^0(\underbrace{X\cup N_y}_{\simeq X};R)\oplus H^0(\underbrace{Y\cup N_x}_{\simeq Y};R)\longrightarrow H^0(\underbrace{N_x\cap N_y}_{\simeq\{*\}};R)\to 0$$

the kernel is the desired subalgebra $\square$

Example 11 $H^\bullet(S^1 \vee S^1 \vee S^2) \cong$

$\mathbb{Z}\langle x_1, x_2, x_3\rangle / (x_i x_j = 0 \text{ for all } i,j)$

$\deg x_1 = \deg x_2 = 1, \ \deg x_3 = 2$

This is not isomorphic to the ring $H^\bullet(S^1 \times S^1)$, which contains elements of degree 1 with non-zero product.

$\Rightarrow S^1 \vee S^1 \vee S^2 \not\simeq S^1 \times S^1$

Theorem 13 X top. space, $A \subseteq X$, $\varphi \in H^n(X, A; R)$, $\psi \in H^k(X, A; R)$. Then

$$\varphi \smile \psi = (-1)^{nk} \psi \smile \varphi$$

Proof: next lecture.

This property of the graded R-alg. $H^\bullet(X, A; R)$ is called graded commutative.