Theorem 13 X top. space, $[\varphi] \in H^n(X;R)$, $[\psi] \in H^k(X;R)$. Then Hatcher Thm 3.11, p. 210

$$[\varphi] \smile [\psi] = (-1)^{nk} [\psi] \smile [\varphi].$$

Proof For $\sigma : \Delta^n \to X$, let $\bar\sigma : \Delta^n \to X$ be $\bar\sigma = \sigma \circ$ (natural homeo $[e_0, \dots, e_n] \to [e_n, e_{n-1}, \dots, e_1, e_0]$), i.e. $\bar\sigma(e_i) = \sigma(e_{n-i})$. Let $\rho : C_\bullet(X) \to C_\bullet(X)$, $\sigma \mapsto (-1)^{\varepsilon_n} \bar\sigma$, where $\varepsilon_n = \frac{(n+1)n}{2}$.

Claim 1: ρ is a chain map.

Claim 2: $\rho \simeq \mathrm{id}_{C_\bullet(X)}$

Pf that Claim 1&2 $\Rightarrow$ Thm:

$$(\rho^*(\varphi \smile \psi))(\sigma) = (-1)^{\varepsilon_{n+k}} \varphi(\sigma|_{[e_{n+k}, \dots, e_k]}) \psi(\sigma|_{[e_k, \dots, e_0]})$$

$$((\rho^*\psi) \smile (\rho^*\varphi))(\sigma) = (-1)^{\varepsilon_n + \varepsilon_k} \psi(\sigma|_{[e_k, \dots, e_0]}) \psi(\varphi|_{[e_{n+k}, \dots, e_k]})$$

$$\Rightarrow [\varphi] \smile [\psi] = [\varphi \smile \psi] = [\rho^*(\varphi \smile \psi)]$$
$$= (-1)^{\varepsilon_{n+k} + \varepsilon_n + \varepsilon_k} [(\rho^*\psi) \smile (\rho^*\varphi)] = (-1)^{nk} [\rho^*\psi] \smile [\rho^*\varphi]$$
$$= (-1)^{nk} [\psi] \smile [\varphi].$$

Check that $\varepsilon_{n+k} + \varepsilon_n + \varepsilon_k \equiv nk \ (2)$ ✓

Pf of Claim 1:
$$\rho\, d\sigma = \rho\left(\sum_{i=0}^n (-1)^i \sigma|_{[e_0, \dots, \hat e_i, \dots, e_n]}\right)$$
$$= \sum_{i=0}^n (-1)^{i + \varepsilon_{n-1}} \sigma|_{[e_n, \dots, \hat e_i, \dots, e_0]}$$

$$d\rho\sigma = \sum_{j=0}^n (-1)^{j + \varepsilon_n} \sigma|_{[e_n, \dots, \widehat{e_{n-j}}, \dots, e_0]} \qquad n-j = i$$
$$= \sum_{i=0}^n (-1)^{n-i+\varepsilon_n} \sigma|_{[e_n, \dots, \hat e_i, \dots e_0]}$$

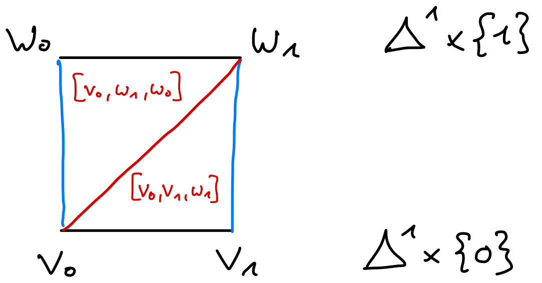
Check: $\varepsilon_{n-1} \equiv n + \varepsilon_n \ (2) \Leftrightarrow n + \frac{n(n-1)}{2} \equiv \frac{n(n+1)}{2}$ ✓

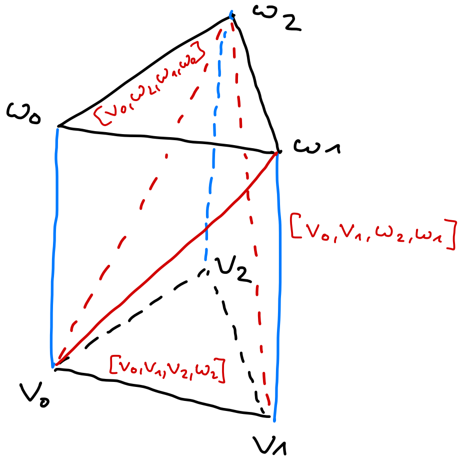
<u>Pf of Claim 2:</u> Need homotopy $s : C_n(X) \to C_{n+1}(X)$ with

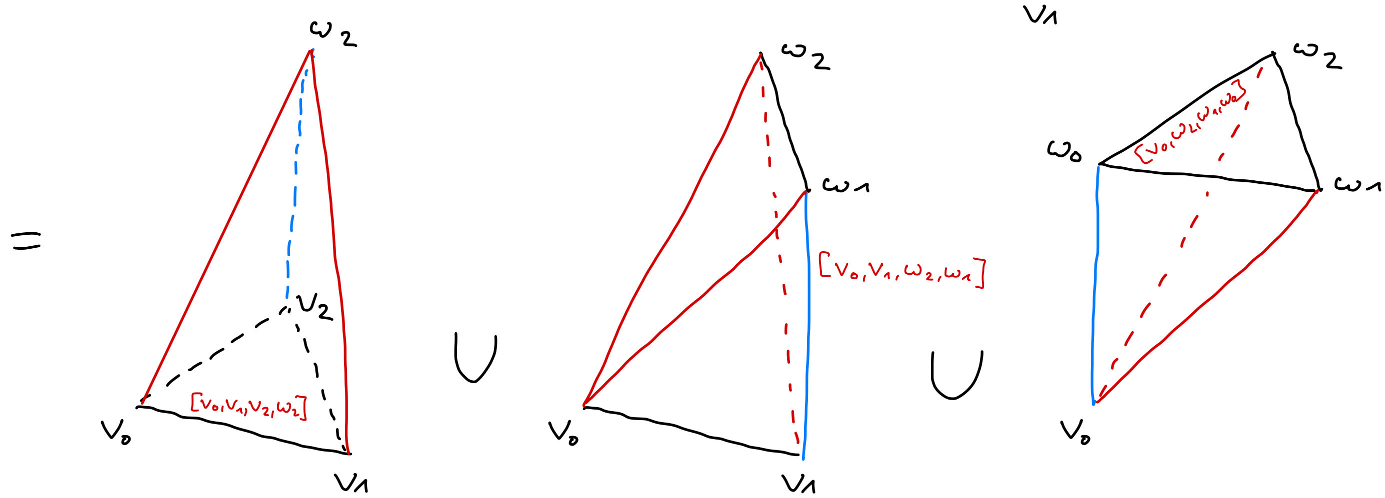
$$d_{n+1} s_n + s_{n-1} d_n = \rho_n - \mathrm{id}_{C_n}. \qquad (*)$$

Construction of s is inspired by the prism operator:
cut the prism $\Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$
into $n+1$ many $(n+1)$-simplices.

Let $v_i = (e_i, 0)$ and $w_i = (e_i, 1)$ for $i = 0, \dots, n$.







Let $\pi : \Delta^n \times [0,1]$ be the projection, so that $\pi(w_i) = \pi(v_i) = e_i$.
Define

$$\boxed{s_n(\sigma) := \sum_{i=0}^{n} (-1)^{i + \varepsilon_{n-i}} \sigma \circ \pi \left([v_0, \dots, v_i, w_n, \dots, w_i] \right)}$$

Let us check by calculation that $(*)$ holds.

$$d_{n+1}(S_n(\sigma)) \overset{(1)}{=} \sum_{0\le j\le i\le n} (-1)^{i+\varepsilon_{n-i}+j} \, \sigma\circ\pi([v_0,\dots,\hat{v}_j,\dots,v_i,w_n,\dots,w_i])$$

$$\overset{(2)}{+} \sum_{0\le i\le j\le n} (-1)^{\varepsilon_{n-i}+n+j+1} \, \sigma\circ\pi([v_0,\dots,v_i,w_n,\dots,\underset{\substack{\uparrow \\ \text{index } n-j+i+1}}{\hat{w}_j},\dots w_i])$$

Consider the summands with $i=j$:

$$(-1)^{\varepsilon_n}\sigma\circ\pi([w_n,\dots,w_0]) +$$

$$+ \sum_{i=1}^{n+1} (-1)^{\varepsilon_{n-i}} \sigma\circ\pi([v_0,\dots,v_{i-1},w_n,\dots,w_i])$$

$$+ \sum_{k=0}^{n} (-1)^{\varepsilon_{n-k}+n+k+1} \sigma\circ\pi([v_0,\dots,v_k,w_n,\dots,w_{k+1}])$$

$$+ (-1)^{\varepsilon_0}\sigma\circ\pi([v_0,\dots,v_n])$$

these cancel: index shift $k=i-1$, check $\varepsilon_{n-i} \not\equiv \varepsilon_{n-i+1} + n + i \ (2)$

$$= (-1)^{\varepsilon_n}\bar{\sigma} + \sigma = \rho\sigma - \sigma$$

So, to prove $(*)$, one has to check that the summands with $i\neq j$ equal $-S_{n-1}(d_n(\sigma))$

$$= -S_{n-1}\Big(\sum_{j=0}^{n}(-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}\Big)$$

$$= \sum_{0\le j\le k\le n} (-1)^{1+j+k+\varepsilon_{n-k-1}} \sigma\circ\pi([v_0,\dots,\hat{v}_j,\dots,v_{k+1},w_n,\dots,w_{k+1}])$$

index shift: $k=i-1$. check $i+\varepsilon_{n-i}+j \equiv 1+j+i-1+\varepsilon_{n-i}$

$\Rightarrow$ equals summands of (1) with $j<i$

$$+ \sum_{0\le i<j\le n} (-1)^{1+j+i+\varepsilon_{n-i-1}} \sigma\circ\pi([v_0,\dots,v_i,w_n,\dots\hat{w}_j,\dots w_i])$$

check: $\varepsilon_{n-i}+n+j+1 \equiv 1+j+i+\varepsilon_{n-i-1}$

$\Rightarrow$ equals summands of (2) with $i<j$

$\square$

Remark 14 We'll prove later that:

$$H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1}) \qquad \text{with } \deg x = 2$$

(commutative since $H^k(\mathbb{C}P^n) = 0$ for odd k)

$$H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1}) \qquad \text{with } \deg x = 1$$

(commutative because of $\mathbb{Z}/2$ coefficients)

$$H^\bullet((S^1)^{\times n}) \cong \mathbb{Z}\langle x_1, \dots, x_n\rangle / (x_i x_j + x_j x_i,\ x_i^2)$$

with $\deg x_i = 1$

(not commutative, but graded commutative)

Reminder from Alg Top 1 X top. space, $A, B \subseteq X$.

$C_n(A+B) \subseteq C_n(A\cup B)$ is generated by $C_n(A) \cup C_n(B) \subseteq C_n(A\cup B)$.

$C_n(A+B)$ is a chain complex, and $C_\bullet(A+B) \overset{i}{\hookrightarrow} C_\bullet(A\cup B)$

is a homotopy equivalence (proved by barycentric subdivision).

Lemma 14 There is a (natural) iso

$H^n(X, A\cup B; R) \xrightarrow{j} H^n(X, A+B; R)$ induced by i.

Proof (skipped in lecture)

$$\begin{tikzcd}
0 \arrow[r] & C_n(A+B) \arrow[r] \arrow[d, "i"] & C_n(X) \arrow[r] \arrow[d, "\mathrm{id}_X"] & C_n(X, A+B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A\cup B) \arrow[r] & C_n(X) \arrow[r] & C_n(X, A\cup B) \arrow[r] & 0
\end{tikzcd}$$

Commutes, has split exact rows. Apply $\mathrm{Hom}(-, R)$ and take the natural LESs in cohomology:

$$\begin{tikzcd}
\cdots & H^n(A+B;R) \arrow[l] & H^n(X;R) \arrow[l] & H^n(X, A+B;R) \arrow[l] & \cdots \arrow[l] \\
\cdots & H^n(A\cup B;R) \arrow[l] \arrow[u, "i^*", "\text{iso}"'] & H^n(X;R) \arrow[l] \arrow[u, "\mathrm{id}"] & H^n(X, A\cup B);R \arrow[l] \arrow[u, "j"] & \cdots \arrow[l]
\end{tikzcd}$$

j is an iso by the five lemma. $\square$

Def Let X be a top. space and $A, B \subseteq X$. Let the **relative cup product**

$$\smile : H^n(X, A; R) \times H^k(X, B; R) \longrightarrow H^{n+k}(X, A \cup B; R)$$

be the postcomposition with j^{-1} of the bilinear map on cohomology induced by

$$\smile : C^n(X, A; R) \times C^k(X, B; R) \longrightarrow C^{n+k}(X, A+B; R)$$

$$(\varphi \smile \psi)(\underset{\substack{\text{im } \sigma \subseteq A \text{ or} \\ \text{im } \sigma \subseteq B}}{\sigma}) = \underbrace{\varphi(\sigma|_{[e_0, \dots, e_n]})}_{0 \text{ if im } \sigma \subseteq A} \underbrace{\psi(\sigma|_{[e_n, \dots, e_{n+k}]})}_{0 \text{ if im } \sigma \subseteq B}$$