I Motivation

Def (Poincaré algebra) A connected ($\Leftrightarrow A^0 = \Bbbk$) gca $A^\bullet = \bigoplus_{j=0}^{\infty} A^j$ over a field $\Bbbk$ is called a Poincaré algebra of formal dimension n if.

(i) $A^j = 0$ for $j > n$.

(ii) $A^n \cong \Bbbk$

(iii) the bilinear pairing $A^j \otimes A^{n-j} \longrightarrow A^n \cong \Bbbk$ is non-degenerate
$\Longleftrightarrow$ the map $A^j \longrightarrow \mathrm{Hom}_{\Bbbk}(A^{n-j}, \Bbbk)$ is an isomorphism.

Claim Let M^n be a closed connected orientable manifold.
Then $H^\bullet(M;\mathbb{Q})$ is a Poincaré algebra of formal dimension n.

II Manifolds

Def (Topological manifold) A Hausdorff second countable topological space M is called a topological manifold (resp. top. mnfd with boundary) of dimension n if each point $x \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (resp. of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$).

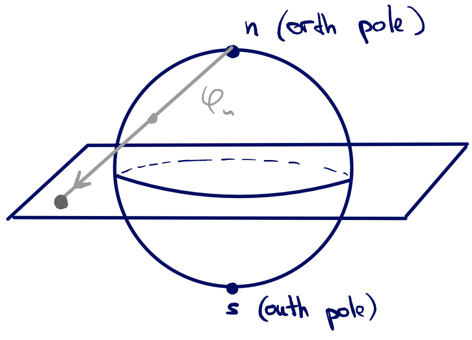
Def (Boundary) Let M be a manifold with boundary. The subset ∂M of points $x \in M$ that do not have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ is called the boundary of M.

Def (Closed manifold) A compact manifold without boundary is called closed.

Examples: (i) $\mathbb{R}^n$ any any open subset of $\mathbb{R}^n$.

(ii) $S^n := \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} (x^i)^2 = 1\}$

Two charts: $\varphi_n : S^n \setminus \{n\} \longrightarrow \mathbb{R}^n$
$$(x^0, \dots, x^n) \longmapsto \left(\frac{x^0}{1-x^n}; \dots; \frac{x^{n-1}}{1-x^n}\right)$$
$\varphi_s : S^n \setminus \{s\} \longrightarrow \mathbb{R}^n$
$$(x^0, \dots, x^n) \longmapsto \left(\frac{x^0}{1+x^n}; \dots; \frac{x^{n-1}}{1+x^n}\right)$$
with transition maps: $\varphi_s \circ \varphi_n^{-1}, \varphi_n \circ \varphi_s^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$.
$$(t^1, \dots, t^n) \longmapsto \left(\frac{t^1}{\|t\|^2}; \dots; \frac{t^n}{\|t\|^2}\right)$$



(iii) n-dimensional torus T^n;

(iv) real and complex projective spaces $\mathbb{RP}^n$ & $\mathbb{CP}^n$.

... **with boundary:** (i) D^n;

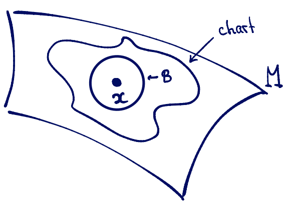
(ii) solid torus $S^1 \times D^2$.

Non-examples: (i) 

(ii) $\mathbb{RP}^\infty = \bigcup_{n=0}^{\infty} \mathbb{RP}^n$ & $\mathbb{CP}^\infty = \bigcup_{n=0}^{\infty} \mathbb{CP}^n$.

Proposition 1. Let M^n be a topological manifold. Then for any $x \in M$: $H_i(M, M \setminus \{x\}; R) \cong \begin{cases} 0 & \text{if } i \neq n; \\ R & \text{if } i = n. \end{cases}$

▷ Let B be an open ball around x (sits inside of a neighborhood of x homeomorphic to a subset of $\mathbb{R}^n$).



$\Longrightarrow Z := M \setminus B$ is closed.

By excision theoreom, $H_i(M, M\setminus\{x\}; R) \cong H_i(M\setminus Z; (M\setminus\{x\})\setminus Z; R) \cong H_i(B, B\setminus\{x\}; R)$

$$\Rightarrow \cdots \to \underset{0}{\underset{\parallel}{\widetilde{H}_i(B;R)}} \to H_i(B, B\setminus\{x\}; R) \xrightarrow{\cong} \underset{\widetilde{H}_{i-1}(S^n;R)}{\underset{\cong}{\widetilde{H}_{i-1}(B\setminus\{x\};R)}} \to \underset{0}{\underset{\parallel}{\widetilde{H}_{i-1}(B;R)}} \to \cdots$$

$\square$

Def (Local (co)homology) $H_i(M, M\setminus\{x\}; R)$ local homology group
$H^i(M, M\setminus\{x\}; R)$ local cohomology group

Def (Homology manifold) A Hausdorff second countable space is a homology R-manifold of dimension n if for any $x \in M$ $H_\bullet(M, M\setminus\{x\}; R) \cong \widetilde{H}_\bullet(S^n; R)$.

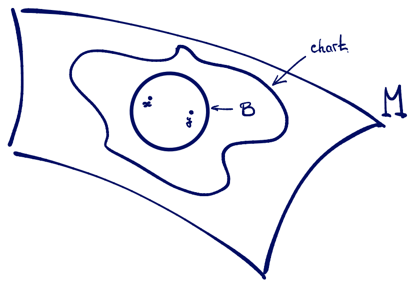
III Orientations

Def (Local orientations) A local orientation μ_x in $x \in M$ is a generator of the local homology group $H_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$.

Note that there are two choices of a generator in $\mathbb{Z}$.
$\Rightarrow$ At each point there are two possible orientations.

Def (Orientation) An orientation of an n-dimensional manifold is a choice of a local orientation $\mu_x \in H_n(M, M\setminus\{x\}; \mathbb{Z})$ at every $x \in M$, s.t. it is locally consistent, i.e. if $x, y \in M$ can be covered by a ball B within one chart. then μ_x and μ_y map one to each other under the isomorphisms:

$$H_n(M, M\setminus\{x\}; \mathbb{Z}) \xleftarrow{\cong} H_n(M, M\setminus B; \mathbb{Z}) \xrightarrow{\cong} H_n(M, M\setminus\{y\}; \mathbb{Z})$$



Def ((non-) Orientable manifold) A manifold is orientable if there exists an orientation on M.
A manifold is non-orientable if it is not orientable.

Examples: (i) S^n is orientable.
(ii) The Möbius band is non-orientable.

Proposition 2 Let M be a closed connected manifold of dimension n.
(i) The homomorphism $H_n(M; \mathbb{F}_2) \to H_n(M, M\setminus\{x\}; \mathbb{F}_2)$ is an isomorphism for any $x \in M$.
(ii) If M is orientable, then $H_n(M; \mathbb{Z}) \to H_n(M, M\setminus\{x\}; \mathbb{Z})$ is an isomorphism for any $x \in M$.
If M is non-orientable, then $H_n(M; \mathbb{Z}) = 0$.
(iii) $H_i(M; \mathbb{Z}) = 0$ for $i > n$.

Main Lemma 3 Let $A \subseteq M$ be a compact subset of a manifold M of dimension n. (not necessary compact).
(i) $H_i(M, M\setminus A; R) = 0$ if $i > n$.
$\alpha \in H_n(M, M\setminus A; R)$ is zero iff its image in $H_n(M, M\setminus\{x\}; R)$ is zero for every $x \in A$.
(ii) For every locally consistent choice of orientations μ_x, $x \in A$, exists a unique $\mu_A \in H_n(M, M\setminus A; R)$ s.t. is μ_x for all $x \in A$.

▷ <u>STEP 1</u>. If the assertion holds for compact A, B and $A\cap B$, then it holds for $A\cup B$.

Relative Mayer-Vietoris sequence:

$$\underset{\substack{\| \\ 0}}{H_{n+1}(M, M\setminus(A\cap B))} \longrightarrow H_n(M, M\setminus(A\cup B)) \xrightarrow{\Phi} H_n(M, M\setminus A)\oplus H_n(M, M\setminus B) \xrightarrow{\Psi} H_n(M, M\setminus(A\cap B))$$

For $i>n$ we have $H_i(M, M\setminus(A\cap B)) = H_i(M, M\setminus A) = H_i(M, M\setminus B) = 0$ $\Longrightarrow$ $H_i(M, M\setminus(A\cup B))$ is locked between two zeros $\Rightarrow$ zero itself.

If $\mu \in H_n(M, M\setminus(A\cup B))$ is s.t. $\mu_x \in H_n(M, M\setminus\{x\})$ is zero for all $x\in A\cup B$ $\Longrightarrow$ its images in $H_n(M, M\setminus A)$ and $H_n(M, M\setminus B)$ are zero by the assumption. $\Longrightarrow$ Since Φ is injective, $\mu = 0$. (Proves (i)).

Let μ_x, $x\in A\cup B$ be a locally consistent choice of orientations $\Longrightarrow$ $\exists!$ $\mu_A \in H_n(M, M\setminus A)$, $\mu_B\in H_n(M, M\setminus B)$

$\Psi(\mu_A, \mu_B) = \mu_A|_{A\cap B} - \mu_B|_{A\cap B} \in H_n(M, M\setminus(A\cap B))$. its image is zero in $H_n(M, M\setminus\{x\})$ for any $x\in A\cap B$

$\Longrightarrow$ it is zero itself by assumption on $A\cap B$. $\Longrightarrow$ By exactness, (μ_A, μ_B) is the image of a unique (since Φ is injective) element $\mu_{A\cup B}\in H_n(M, M\setminus(A\cup B))$. $\boxed{1}$

<u>STEP 2</u>. It is enough to prove the assertion for a compact subset of a single chart. (i.e. in $\mathbb{R}^n$)

Any compact subset $A\subseteq M$ is a union of a finite number of compact subsets, s.t. each belongs to a chart $\Longrightarrow$ We can apply induction and <u>STEP 1</u>.
If U is a chart, then $H_i(M, M\setminus A)\cong H_i(U, U\setminus A)$ by excision. $\boxed{2}$

$\Rightarrow$ From now on we assume $M=\mathbb{R}^n$.

<u>STEP 3</u> If $A\subseteq\mathbb{R}^n$ is a finite simplicial complex, s.t. its simplices are linearly embedded, then the assertion follows by induction, and it is enough to prove for one simplex. The latter follows from the definition of local consistency. $\boxed{3}$

<u>STEP 4</u>. $A\subseteq\mathbb{R}^n$ compact

$\alpha\in H_i(\mathbb{R}^n, \mathbb{R}^n\setminus A)$ is represented by a relative cycle z and let $C\subseteq\mathbb{R}^n\setminus A$ be a union of the images of the singular simplices of ∂z.

A and C are compact $\Longrightarrow$ they have positive distance $\delta>0$ between them.

Cover A with a finite piecewise linear simplicial complex K with $K\cap C=\emptyset$: (i) cover A by one big enough simplex;
from Step 3
(ii) take barycentric subdivision s.t. the diameter of a piece is less than δ.
(iii) take simplices that intersect A.

The same chain z represents a class $\alpha_K\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus K)$ that maps to $\alpha\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus A)$.
By <u>STEP 3</u>, $\alpha_K = 0$ for $i>n$ $\Longrightarrow$ $\alpha = 0$ and $H_i(\mathbb{R}^n, \mathbb{R}^n\setminus A) = 0$ for $i>n$.

Finally, assume $i=n$. If $\alpha_{K,x} = 0\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\})$ for all $x\in A$, then it also holds for all $x\in K$.
Indeed, for any simplex $\Delta\in K$ and any $x\in\Delta$ the map $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\Delta)\longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\})$ is an iso.
<u>STEP 3</u> now implies that $\alpha_K = 0$ $\Longrightarrow$ $\alpha = 0$, which concludes the proof of (i) and uniqueness part in (ii).

<u>Existence</u>: let $\alpha_A\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus A)$ be the image of $\alpha_B\in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B)$, where B is a big ball containing A. $\square$
exists by definition of local consistency.