Last time: Proof of

Lemma 3 M^n without boundary, $A \subseteq M$ compact, R commutative unital ring.

(i) $H_i(M, M \setminus A; R) = 0$ for $i > n$.

$\alpha \in H_n(M, M\setminus A; R)$ is zero $\Longleftrightarrow$

image of α in $H_n(M, M \setminus \{x\}; R)$ is zero for all $x \in A$.

(ii) μ_x locally consistent choice of orientation for $x \in A$

$\Rightarrow$ exists unique $\mu_A \in H_n(M, M\setminus A; R)$ mapping to μ_x for all $x \in A$

Today:

Prop 2 M^n closed ($\Longleftarrow$ compact, no boundary) connected.

(i) $H_n(M; \mathbb{F}_2) \to H_n(M, M\setminus\{x\}; \mathbb{F}_2)$ iso for all $x \in M$.

(ii) M orientable $\Rightarrow$ $H_n(M; \mathbb{Z}) \to H_n(M, M\setminus\{x\}; \mathbb{Z})$ iso $\forall x \in M$.

M non-orientable $\Rightarrow$ $H_n(M; \mathbb{Z}) = 0$.

(iii) $H_i(M; \mathbb{Z}) = 0$ for $i > n$.

Note that (iii) follows from Lemma 3 (i) with $A = M$. For (i) & (ii), we'll also use Lemma 3, but need some more tools.

For M^n without boundary, let Hatcher p. 235

$\tilde{M} := \{ \mu_x \mid x \in M$ and $\mu_x \in H_n(M, M\setminus\{x\})$ a local orientation$\}$

Note $p: \tilde{M} \to M$, $\mu_x \mapsto x$ is a $2:1$ surjection. For $B \subseteq$ chart $\subseteq M$ an open ball and a generator $\mu_B \in H_n(M, M\setminus B)$, let

$U_{(\mu_B)} := \{ \mu_x \in \tilde{M} \mid x \in B$, μ_x image of μ_B under $H_n(M, M\setminus B) \to H_n(M, M\setminus\{x\})\}$

Exercise The $U_{(\mu_B)}$ form the base of a topology on $\tilde{M}$, st p is a $2:1$ covering.

Def $p: \tilde{M} \to M$ is called the orientation covering of M.

Each $\mu_x \in \tilde{M}$ has a canonical orientation $\tilde{\mu}_x \in H_n(\tilde{M}, \tilde{M} \setminus \mu_x)$ corresponding to μ_x under the isos

$$H_n(\tilde{M}, \tilde{M} \setminus \mu_x) \xleftarrow[\text{excision}]{} H_n(U_{(\mu_B)}, U_{(\mu_B)} \setminus \mu_x)$$
$$\longrightarrow H_n(B, B \setminus x) \xrightarrow[\text{excision}]{} H_n(M, M \setminus x)$$

These are locally consistent, so $\tilde{M}$ has a canonical orientation.

Prop 4 If M is connected, then: $\tilde{M}$ non-connected $\Leftrightarrow$ M orientable

Proof M has orientation $\mu_x \Rightarrow \tilde{M} = \underbrace{\{\mu_x \mid x \in M\}}_{\text{open}} \sqcup \underbrace{\{-\mu_x \mid x \in M\}}_{\text{open}}$

If $\tilde{M}$ has two components N_1, N_2, then they inherit an orientation from $\tilde{M}$. Check that $p|_{N_i} : N_i \to M$ are coverings. Then, they must be one-sheeted coverings, i.e. homeomorphisms. $\square$

Example $\widetilde{S^2} \cong S^2 \sqcup S^2$, $\widetilde{\mathbb{RP}^2} \cong S^2$, $\widetilde{\text{Klein Bottle}} \cong S^1 \times S^1$

Note that $S^3 \to \mathbb{RP}^3$ is an orientable double covering, but not the orientation covering, which is $\mathbb{RP}^3 \sqcup \mathbb{RP}^3 \to \mathbb{RP}^3$ (since $\mathbb{RP}^3$ is orientable).

Def A section of p is a cont. map $s : M \to M_R$ with $ps = id_M$.

Note that a section of a covering map has a component of M as image

Prop 5 μ_x is an orientation $\Leftrightarrow$ $x \mapsto \mu_x$ is a section of p

Pf Exercise $\square$

Def R commutative unital ring, M^n without boundary.

Local R-orientation: μ_x is a generator of $H_n(M, M \setminus x; R)$

R-orientation: locally consistent choice of local R-orientations.

M R-orientable $:\Leftrightarrow$ There exists an R-orientation

Example Every M is $\mathbb{F}_2$-orientable, since there is precisely one local $\mathbb{F}_2$-orientation at every point.

Def Let $M_R := \{\alpha_x \mid x \in M,\ \alpha_x \in H_n(M, M\setminus\{x\}; R)\}$, with similar topology as $\tilde{M}$.

Note $p_R : M_R \longrightarrow M$ is an $|R|$-sheeted covering.

Prop 6 Let $M_r = \{\alpha_x \mid \alpha_x$ is the image of $\mu_x \otimes r$ under the iso
$$H_n(M, M\setminus x) \otimes R \longrightarrow H_n(M, M\setminus x; R)$$
for μ_x a generator of $H_n(M, M\setminus x)\}$

Then: $M_r \subseteq M_R$ is open; $M_r = M_{-r}$;
$M_r \cap M_s = \emptyset$ for $r \neq \pm s$;
$M_r \cong M$ if $r = -r$, and $M_r \cong \tilde{M}$ if $r \neq -r$.

Pf: Exercise $\square$

Prop 7 μ_x is an R-orientation $\Leftrightarrow$
$x \longmapsto \mu_x$ is a section of p_R with each μ_x a generator of $H_n(M, M\setminus x; R)$

Pf Exercise, similar to Prop 5. $\square$

Prop 8 If $0 = 2$ in R $\Rightarrow$ all M^n are R-orientable
If $0 \neq 2$ in R $\Rightarrow$ M^n is R-orientable iff it is $\mathbb{Z}$-orientable

Proof $0 = 2 \Rightarrow M_1 \cong M \Rightarrow p_R$ has a section to $M_1 \Rightarrow M$ is R-orientable

Assume $0 \neq 2$. Generators of $H_n(M, M\setminus x; R)$ are of the form $\mu_x \otimes u$ for μ_x a gen. of $H_n(M, M\setminus x)$ and $u \in R$ a unit. Then $u \neq -u$ $\Rightarrow M_u \cong \tilde{M} \Rightarrow p_R$ has a section to M_u iff $\tilde{M} \to M$ has a section. $\square$

Proof of Prop 2 (i) and (ii) Pointwise sum and pointwise R-multiplication turn $\Gamma(M, M_R)$ into an R-module.

$$H_n(M;R) \longrightarrow \Gamma(M, M_R),$$
$$\alpha \mapsto (x \mapsto \text{image of } \alpha \text{ in } H_n(M, M\setminus x; R))$$

is a homomorphism. By Lemma 3, applied to $A = M$, it is an isomorphism! Indeed, Lemma 3 (i) yields injectivity. And Lemma 3 (ii) yields surjectivity (here, we need a slightly more general version of Lemma 3(ii): namely, for every locally consistent choice $\alpha_x \in H_n(M, M\setminus x; R)$, $\exists!\, \mu_A \in H_n(M, M\setminus A; R)$ that maps to α_x for all x. The proof is the same — we never use that α_x generates).

$$M \ R\text{-orientable} \Rightarrow \left\{ \begin{array}{ll} \tilde{M} = M \sqcup M & \text{if } 0 \neq 2 \\ M_r = M \text{ for all } r \in R & \text{if } 0 = 2 \end{array} \right\} \Rightarrow M_R \cong \coprod_{r \in R} M$$

$\Rightarrow \Gamma(M, M_R) \cong R$ (using connectedness of M) $\Rightarrow H_n(M;R) \cong R$.

So $H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$ for all M (since all M are $\mathbb{F}_2$-orientable), and $H_n(M) \cong \mathbb{Z}$ for all orientable M.

M non-orientable $\Rightarrow \tilde{M}$ is connected $\Rightarrow$

$$M_{\mathbb{Z}} \cong \underbrace{M_0}_{\cong M} \sqcup \underbrace{M_1}_{\cong \tilde{M}} \sqcup \underbrace{M_2}_{\cong \tilde{M}} \cdots$$

So the only section of $p_{\mathbb{Z}}$ goes to $M_0 \Rightarrow \Gamma(M, M_{\mathbb{Z}}) \cong 0$

$\Rightarrow H_n(M) \cong 0$. $\square$