Corollary 9 (i) Let M be a closed R-oriented n-manifold. Then there exists a unique class $\mu \in H_n(M;R)$ st for all $x \in M$, the isom $H_n(M, M \setminus \{x\}; R)$ sends μ to the given local orientation.

(ii) If M is connected, then μ generates $H_n(M;R) \cong R$.

Proof (i) directly from Lemma 3, (ii) similar to Prop 2. $\square$

Def The class from Corollary 9 is called the **fundamental class** of M, written $[M]_R \in H_n(M;R)$.

drop R from notation for $R = \mathbb{Z}$.

Remark 10 If M^n is closed and has a Δ-complex structure, then:

(1) Every simplex of M is a subsimplex of an n-simplex.

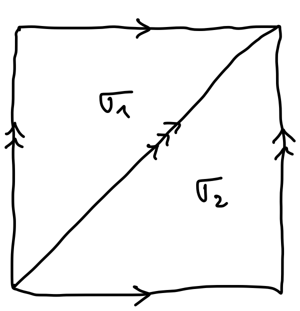
(2) Every $(n-1)$-simplex is a face of precisely two n-simplexes.

(3) M has only finitely many n-simplexes $\sigma_1, \dots, \sigma_k$.

If M is oriented, then $[M] = \left[\sum_{i=1}^{k} \varepsilon_i \sigma_i\right]$ with $\varepsilon_i = \pm 1$.

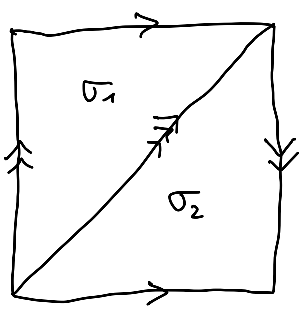
such that in $\sum_{i=1}^{k} \varepsilon_i d\sigma_i$, each $(n-1)$-simplex appears once with $+$, once with $-$. If M is not orientable, no such choice of ε_i exists. Over $\mathbb{F}_2$, $[M]_{\mathbb{F}_2} = \left[\sum_{i=1}^{k} \sigma_i\right]$.

For example:



Torus T

$[T] = \pm[\sigma_1 - \sigma_2]$



Klein bottle K

$[K]_{\mathbb{F}_2} = [\sigma_1 + \sigma_2]$

Def M^n, N^n compact, oriented, connected, $f: M \longrightarrow N$ continuous.

Then the degree of f is the unique integer $\deg f$ st

$$f_*([M]) = \deg f \cdot [N] \in H_n(N).$$

For not necessarily orientable M, N, there is a unique $\deg_{\mathbb{F}_2} f \in \mathbb{F}_2$ st

$$f_*([M]_{\mathbb{F}_2}) = \deg_{\mathbb{F}_2} f \cdot [N]_{\mathbb{F}_2} \in H_n(N; \mathbb{F}_2)$$

This extends our previous def of deg for $f: S^n \longrightarrow S^n$.

Remark 11 $\deg f \circ g = \deg f \cdot \deg g$ easily follows.

Theorem 12 (Hopf 1927)

$f, g: M^n \longrightarrow S^n$ for M compact, connected, oriented. Then:

$$f \simeq g \iff \deg f = \deg g.$$

Conjecture 13 (Hopf 1931)

$f: M^n \longrightarrow M^n$ for M compact, connected, oriented Then:

$$f \simeq \mathrm{id}_M \iff \deg f = 1$$

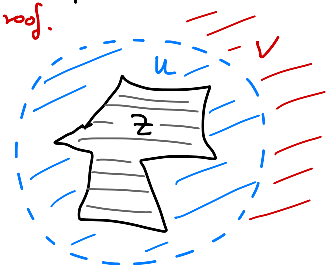
Proposition 14 M^n non-compact and connected

$\Rightarrow H_i(M;R)=0$ for all $i \geq n$.

Proof Let $[z] \in H_i(M)$. To show: $[z]=0$. Pick $U \subseteq M^n$ open st

↳ Drop the R from notation in this proof.

$\mathrm{im}(z) \subseteq U$ and $\overline{U}$ compact. Let $V = M \setminus \overline{U}$.



Consider the LES of $(M, U\cup V, V)$:

$$\begin{tikzcd}
H_{i+1}(M, U\cup V) \arrow[r, "\partial"] & H_i(U\cup V, V) \arrow[r, "\mathrm{incl}_*"] & H_i(M, V) \\
& H_i(U) \arrow[u, "\text{excision}", "\cong"'] \arrow[r, "\mathrm{incl}_*"'] & H_i(M) \arrow[u, "\mathrm{incl}_*"']
\end{tikzcd}$$

<u>$i \geq n+1$</u> $\Rightarrow$ top left & right term zero by Lemma 3 $\Rightarrow$ top middle zero $\Rightarrow$

$H_i(U)=0 \Rightarrow [z]=0 \in H_i(U) \Rightarrow [z]=0 \in H_i(M)$. ✓

<u>$i=n$</u> $[z]$ defines a section $M \to M_R$ by

$x \longmapsto (x, \text{ image of } [z] \text{ under } H_n(M) \to H_n(M \setminus x))$

Pick $x_0 \in V$. Then $x_0 \mapsto (x_0, 0)$. M connected $\Rightarrow$

$\exists$ unique section $M \to M_R$ with $x_0 \mapsto (x_0, 0)$ $\Rightarrow$ the section

defined by $[z]$ sends $x \mapsto (x,0)$ for all x.

Lemma 3 $\Rightarrow [z]=0 \in H_i(M,V)$. Top left term zero $\Rightarrow$

$[z]=0 \in H_i(U\cup V, V) \Rightarrow [z]=0 \in H_i(U) \Rightarrow [z]=0 \in H_i(M)$ $\square$

(8) Poincaré Duality

Sneak preview

Theorem 4 (Poincaré duality)
Let M be a closed R-oriented n-dim manifold. Then for all $k \in \mathbb{Z}$,
$$H^k(M;R) \cong H_{n-k}(M;R).$$

Theorem 7 M^n compact (potentially with ∂) $\Rightarrow$ $H_\bullet(M;R)$ is a finitely generated R-module.

Proof idea Use that $M \simeq$ some finite Δ-complex (Hatcher A.8, A.9 p. 527) $\square$

Corollary 8 M^n closed, $\mathbb{k}$-orientable for a field $\mathbb{k}$
$$H_k(M;\mathbb{k}) \underset{\text{UCT}}{\cong} H^k(M;\mathbb{k}) \underset{\text{PD}}{\cong} H_{n-k}(M;\mathbb{k})$$

Corollary 9 M^n closed, n odd $\Rightarrow \chi(M) = 0$.