Def Let X be a top. space, R a commutative unital ring, $\sigma \in C_n(X;R)$, $\varphi \in C^k(X;R)$ with $k \leq n$.

Then the **cap product** is

$$\sigma \frown \varphi = \varphi(\sigma|_{[e_0,\dots,e_k]})\, \sigma|_{[e_k,\dots,e_n]} \in C_{n-k}(X;R)$$

Proposition 1

(1) Linear extension gives an R-bilinear map

$$C_n(X;R) \times C^k(X;R) \longrightarrow C_{n-k}(X;R)$$

(2) $\sigma \frown \varepsilon = \sigma$ for $\varepsilon \in C^0(X;R)$, $\varepsilon(\tau) = 1 \ \forall \tau$.

(3) $(\sigma \frown \varphi) \frown \psi = \sigma \frown (\varphi \smile \psi)$.

Pf Exercise ☐

Proposition 2 $(-1)^k d(\sigma \frown \varphi) = (d\sigma) \frown \varphi - \sigma \frown d\varphi$

Pf $$d(\sigma \frown \varphi) = \sum_{i=k}^{n} \varphi(\sigma|_{[e_0,\dots,e_k]}) (-1)^{i+k} \sigma|_{[e_k,\dots,\hat{e}_i,\dots e_n]}$$

$$(d\sigma) \frown \varphi = \sum_{j=0}^{k} (-1)^j \varphi(\sigma|_{[e_0,\dots,\hat{e}_j,\dots,e_{k+1}]})\, \sigma|_{[e_{k+1},\dots,e_n]}$$

$$+ \sum_{l=k+1}^{n} (-1)^l \varphi(\sigma|_{[e_0,\dots,e_k]})\, \sigma|_{[e_k,\dots,\hat{e}_l,\dots,e_n]}$$

$$\sigma \frown (d\varphi) = \sum_{m=0}^{k+1} (-1)^m \varphi(\sigma|_{[e_0,\dots,\hat{e}_m,\dots,e_{k+1}]})\, \sigma|_{[e_{k+1},\dots,e_n]}$$

☐

Proposition 3 (1) cycle $\frown$ cocycle $=$ cycle

(2) boundary $\frown$ cocycle $=$ boundary

(3) cycle $\frown$ coboundary $=$ boundary

(4) For $[c] \in H_n(X;R)$, $[\varphi] \in H^k(X;R)$,

$$[c] \frown [\varphi] := [c \frown \varphi] \in H_{n-k}(X;R)$$

is a well-defined R-bilinear map.

(5) X path-connected, $\delta : H_0(X;R) \longrightarrow R$ the iso $[\sigma] \longmapsto 1$, $[c] \in H_n(X;R)$, $[\varphi] \in H^n(X;R)$, then

$$\delta([c] \frown [\varphi]) = \varphi(c) = ev([\varphi])([c])$$

Proof: Exercise.

Theorem 4 (Poincaré duality)

Let M be a closed R-oriented n-dim manifold. Then for all $k \in \mathbb{Z}$,

$$PD : H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$PD([\varphi]) = [M] \frown [\varphi]$$

is an isomorphism.

Before we dive into the consequences of PD, here are two more properties of the cap product.

Prop 5 (Naturality of cap) $f: X \to Y$ cont., $a \in C_n(X)$, $\varphi \in C^k(Y)$

$$f_c(a \frown f^c \varphi) = (f_c a) \frown \varphi$$

Proof Exercise.

Remark 6 Similarly as for the cup, one may define a relative cap

$$\frown : H_n(X, A\cup B; R) \times H^k(X, A; R) \longrightarrow H_{n-k}(X, B; R)$$

using that $C_\bullet(A+B) \hookrightarrow C_\bullet(A\cup B)$ induces isos $H_\bullet(X, A+B) \longrightarrow H_\bullet(X, A\cup B)$.

We'll prove PD, but first, let us harvest some implications.

Let us take the following for granted.

Theorem 7 M^n compact (potentially with ∂) $\Rightarrow$ $H_\bullet(M; R)$ is a finitely generated R-module.

Proof idea Use that $M \simeq$ some finite Δ-complex (Hatcher A.8, A.9 p. 527) $\square$

Corollary 8 M^n closed, $\mathbb{K}$-orientable for a field $\mathbb{K}$

$$H_k(M; \mathbb{K}) \cong H^k(M; \mathbb{K}) \cong H_{n-k}(M; \mathbb{K}) \cong H^{n-k}(M; \mathbb{K})$$

Proof Since $H_\bullet(M)$ f.g. by Thm 7:

$$\dim H_k(M; \mathbb{K}) \overset{\substack{\text{UCT}\\ \text{hom}}}{=} \#\,\mathbb{Z}\text{-summands of } H_k(M) \; +$$

$p = \operatorname{char} \mathbb{K}$ $\qquad \#\,\mathbb{Z}_{p^r}\text{-summands of } H_k(M) \text{ and } H_{k-1}(M)$

$$\overset{\substack{\text{UCT}\\ \text{cohom}}}{=} \dim H^k(M; \mathbb{K})$$

This proves the first iso. The second is PD. $\square$

Corollary 9 M^n closed, n odd $\Rightarrow \chi(M) = 0$.

Proof $\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H_k(M; \mathbb{F}_2)$ $\qquad n = 2m+1$

$$= \sum_{k=0}^{m} (-1)^k \dim H_k(M; \mathbb{F}_2) + (-1)^{2m+1-k} \dim H_{2m+1-k}(M; \mathbb{F}_2) = 0 \quad \square$$

Proposition 10 M^n connected, closed, oriented st $H_\bullet(M)$ is free.

Then $\smile : H^k(M) \times H^{n-k}(M) \longrightarrow H^n(M) \underset{PD}{\cong} H_0(M) \underset{\delta : [\sigma] \mapsto 1}{\cong} \mathbb{Z}$

is **non-singular**, ie

$$\begin{aligned} H^k(M) &\longrightarrow \mathrm{Hom}(H^{n-k}(M), \mathbb{Z}) \\ [\varphi] &\longmapsto ([\psi] \longmapsto \delta(PD([\psi] \smile [\varphi]))) \end{aligned}$$

is an iso.

Proof $H_\bullet(M)$ free by assumption $\Rightarrow$ $\mathrm{Ext}(H_{k-1}(M), \mathbb{Z})$ is trivial

$\Rightarrow$ ev is iso. So we have isos

$$H^k(M) \xrightarrow{ev} \mathrm{Hom}(H_k(M), \mathbb{Z}) \xrightarrow{PD^*} \mathrm{Hom}(H^{n-k}(M), \mathbb{Z})$$

Just need to check that their composition equals the desired homomorphism $[\varphi] \longmapsto ([\psi] \longmapsto \delta(PD([\varphi] \smile [\psi])))$.

Let $[\varphi] \in H^k(M)$, $[\psi] \in H^{n-k}(M)$. Then

$$\begin{aligned} PD^*(ev([\varphi]))([\psi]) &= ev([\varphi])(PD([\psi])) \\ &= ev([\varphi])([M] \frown [\psi]) \\ &= \delta(([M] \frown [\psi]) \frown [\varphi]) \\ &= \delta([M] \frown ([\psi] \smile [\varphi])) \\ &= \delta(PD([\psi] \smile [\varphi])) \end{aligned}$$

$\square$

Remark 11 (1) M^n closed, orientable, $H_\bullet(M)$ free

$\Rightarrow H^k(M) \cong H_k(M) \cong H^{n-k}(M) \cong H_{n-k}(M)$.

(2) A bilinear form $b: \mathbb{Z}^m \times \mathbb{Z}^m \longrightarrow \mathbb{Z}$ is non-singular

$\Leftrightarrow$ $\mathbb{Z}^m \longrightarrow \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z})$, $x \mapsto (y \mapsto b(x,y))$ is an iso

$\Leftrightarrow$ $\forall$ primitive $x \in \mathbb{Z}^m$ (i.e. x not divisible by integers ≥ 2, or equivalently: x can be extended to a basis) $\exists\, y \in \mathbb{Z}^m$ st $b(x,y) = 1$.