Theorem 12 $H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $\deg x = 2$.

Proof By induction over n. For $n=0$, $\mathbb{C}P^0 \cong \{*\}$, $H^\bullet(\{*\}) \cong \mathbb{Z}$

For $n=1$, $\mathbb{C}P^1 \cong S^2$ and $H^\bullet(S^2) \cong \mathbb{Z}[x]/(x^2)$. Assume $n \geq 2$

and $H^\bullet(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n)$. The embedding $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$

induces isos on H^k for $k < 2n$ (evident from CW-structure).

Let x be a generator of $H^2(\mathbb{C}P^n)$. By naturality of $\smile$ and the induction hypothesis, x^k generates $H^{2k}(\mathbb{C}P^n)$ for $k < 2n$.

It just remains to show that x^n generates $H^{2n}(\mathbb{C}P^n)$.

Since $\smile$ is non-singular (Prop 10) and x^k is primitive (since it is a generator), by Rmk 11 (2) $\Rightarrow$ $\exists\, y \in H^{2n-2}(\mathbb{C}P^n)$ st $x \smile y$ generates $H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$. Since $H^{2n-2}(\mathbb{C}P^n) = \mathbb{Z}x^{n-1}$ $\Rightarrow$ $\exists\, m \in \mathbb{Z}$ with $y = m x^{n-1}$. Since $x \smile y = m x^n$ generates $H^{2n}(\mathbb{C}P^n)$ $\Rightarrow$ $m = \pm 1$ $\Rightarrow$ $y = \pm x^{n-1}$ $\Rightarrow$ x^n generates $H^{2n}(\mathbb{C}P^n)$. $\square$

Remark 13 Note that $[\varphi] \in TH^k(X)$

$\Rightarrow$ for all $[\psi] \in H^\ell(X)$ we have $[\varphi] \smile [\psi] \in TH^{k+\ell}(X)$.

So $\smile$ induces $FH^k(X) \times FH^\ell(X) \longrightarrow FH^{k+\ell}(X)$.

recall: $FA = A/TA$ is the "free part" of an ab. group A.

For M closed, connected, oriented,

$$\smile : FH^k(X) \times FH^{n-k}(X) \longrightarrow FH^n(X) \cong \mathbb{Z}.$$

is non-singular (similar proof as for Prop 10).

Proposition 14 (EV for other rings)

Let $C_\bullet$ be a chain complex, R a commutative unital ring, and M an R-module.

(1) There is an isomorphism of cochain complexes over R

$$i : \mathrm{Hom}_{\mathbb{Z}}(C_\bullet, M) \longrightarrow \mathrm{Hom}_R(C_\bullet \otimes_{\mathbb{Z}} R, M)$$
$$\varphi \longmapsto (c \otimes r \longmapsto \varphi(c) r)$$

with inverse i^{-1}:

$$(c \mapsto \psi(c \otimes 1)) \longleftarrow \psi.$$

(2) $$ev_R : H^n(C; M) \longrightarrow \mathrm{Hom}_R(H_n(C; R), M)$$
$$[\varphi] \longmapsto ([\alpha] \longmapsto i(\varphi)(\alpha))$$

is a well-defined R-linear map.

(3)
$$\begin{tikzcd} H^n(C; M) \arrow[rr, "ev"] \arrow[dr, "ev_R"'] & & \mathrm{Hom}_{\mathbb{Z}}(H_n(C), M) \\ & \mathrm{Hom}_R(H_n(C; R), M) \arrow[ur] & \end{tikzcd}$$

$f \longmapsto ([\alpha] \longmapsto f([\alpha \otimes 1_R]))$

commutes.

(4) If R is a field, then ev_R is an isomorphism.

Proof (1) To check: * $i_n(\varphi)$ is an R-homom. $C_n \otimes R \longrightarrow M$

* i_n is an R-homom. at each homological degree
* i is a cochain map
* i_n^{-1} is an $\mathbb{Z}$-homom. $C_n \longrightarrow M$
* $i \circ i^{-1}$, $i^{-1} \circ i$ are identity maps.

(2) To check:

$$i(\varphi)(\alpha) = 0 \text{ if } \begin{cases} \alpha \text{ boundary}, \ \varphi \text{ is cocycle} & \text{or} \\ \alpha \text{ is cycle}, \ \varphi \text{ is coboundary} \end{cases}$$

(3) By def of ev and ev_R.

(4) Same proof as UCT, using Ext^1_R is always 0 since all R-modules are free. $\square$

Prop 15 M^n closed, connected, $\mathbb{K}$-oriented for a field $\mathbb{K}$.
Then $H^\bullet(M;\mathbb{K})$ is a Poincaré algebra of formal dim. n.

Proof (i) $H^j(M;\mathbb{K}) = 0$ for $j > n$
since $H^j(M;\mathbb{K}) \cong H_{n-j}(M;\mathbb{K}) \cong 0$ since $n-j<0$. ✓

(ii) $H^n(M;\mathbb{K}) \cong \mathbb{K}$ since $H^n(M;\mathbb{K}) \xrightarrow[\cong]{PD} H_0(M;\mathbb{K}) \xrightarrow[\cong]{\delta} \mathbb{K}$. ✓

(iii) The $\mathbb{K}$-bilinear pairing

$$\smile : H^j(M;\mathbb{K}) \times H^{n-j}(M;\mathbb{K}) \to H^n(M;\mathbb{K}) \cong \mathbb{K}$$

is non-singular $\iff$ the adjoint homom.

$$H^j(M;\mathbb{K}) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(H^{n-j}(M;\mathbb{K}), \mathbb{K})$$
$$[\varphi] \longmapsto ([\psi] \longmapsto \delta(PD([\varphi]\smile[\psi])))$$

is an iso. Show (similarly as in Prop 10) that the adjoint equals the composition of

$$H^j(M;\mathbb{K}) \xrightarrow{ev_{\mathbb{K}}} \mathrm{Hom}_{\mathbb{K}}(H_j(M;\mathbb{K}),\mathbb{K})$$
$$\xrightarrow{PD^*} \mathrm{Hom}_{\mathbb{K}}(H^{n-j}(M;\mathbb{K}),\mathbb{K})$$ □

Corollary 16 $H^\bullet(\mathbb{R}P^n;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n)$ with $\deg x = 1$.

Proof Same as Thm 12, using Prop 15. □

Long Example 17 M^4 closed, simply connected.

<u>What do we know about $H_\bullet(M), H^\bullet(M)$?</u>

Simply connected $\Rightarrow$ connected $\Rightarrow$ $H_0 \cong H^0 \cong \mathbb{Z}$

—— " —— $\Rightarrow$ orientable $\Rightarrow$ $H_4 \cong H^4 \cong \mathbb{Z}$ and PD holds

—— " —— $\Rightarrow$ $H_1 = 0$ by Hurewicz Thm $\Rightarrow$ $H^3 = 0$ by PD

UCT $\Rightarrow$ $H^1 \cong FH_1 \oplus TH_0 \cong 0$. PD $\Rightarrow$ $H_3 \cong 0$.

UCT $\Rightarrow$ $H^2 \cong FH_2 \oplus TH_1 \cong FH_2$, so H^2 is torsion free and thus free (because $H_\bullet$ f.g. by Thm 7). PD $\Rightarrow$ $H_2 \cong H^2$.

So $H_\bullet(M), H^\bullet(M)$ are determined except for rk $H_2(M) \in \{0,1,2,\dots\}$

<u>What about the cohomology ring?</u> $\smile : H^2(M) \times H^2(M) \longrightarrow H^4(M)$ is non-singular (Prop 10) and symmetric (since $[c_1] \smile [c_2] = (-1)^{2\cdot 2} [c_2] \smile [c_1]$). Pick an orientation of M: that yields an isomorphism $H^4(M) \longrightarrow \mathbb{Z}$ (via $H^4 \xrightarrow{PD} H_0 \xrightarrow{\varepsilon} \mathbb{Z}$)

Pick a basis for $H^2(M)$, ie an iso $H^2(M) \cong \mathbb{Z}^m$. Then $\smile$ becomes a non-singular symmetric bilinear form $\mathbb{Z}^m \times \mathbb{Z}^m \longrightarrow \mathbb{Z}$.

Such a form may be written as a matrix $A \in \mathbb{Z}^{m\times m}$ with

$$v \smile w = v^t A w \quad \text{for } v, w \in \mathbb{Z}^m.$$

Eg for $M = \mathbb{C}P^2$, we find $A = (1)$ or $A = (-1)$, depending on the orientation on $\mathbb{C}P^2$.

$\smile$ Non-singular $\Rightarrow$ $\det A = \pm 1$.

$\smile$ Symmetric $\Rightarrow$ $A^t = A$. Picking a different basis for $H^2(M)$ transforms A into $T^t A T$ for $T \in \mathbb{Z}^{m\times m}$ with $\det T = \pm 1$.

Picking the opposite orientation for M transforms A into $-A$.