Long Example 17 (cont.'d) M^4 closed, simply connected.

Shown last time: $H_0 \cong H_4 \cong \mathbb{Z}$, $H_1 \cong H_3 = 0$, $H_2 = \mathbb{Z}^m$ for some $m \geq 0$.

What about the cohomology ring? $\smile : H^2(M) \times H^2(M) \longrightarrow H^4(M)$

is non-singular (Prop 10) and symmetric (since

$[c_1] \smile [c_2] = (-1)^{2\cdot 2} [c_2] \smile [c_1]$). Pick an orientation of M:

that yields an isomorphism $H^4(M) \longrightarrow \mathbb{Z}$ (via $H^4 \xrightarrow{PD} H_0 \xrightarrow{\varepsilon} \mathbb{Z}$)

Pick a basis for $H^2(M)$, ie an iso $H^2(M) \cong \mathbb{Z}^m$. Then $\smile$ becomes

a non-singular symmetric bilinear form $\mathbb{Z}^m \times \mathbb{Z}^m \longrightarrow \mathbb{Z}$.

Such a form may be written as a matrix $A \in \mathbb{Z}^{m\times m}$ with

$$v \smile w = v^t A w \quad \text{for } v, w \in \mathbb{Z}^m.$$

Eg for $M = \mathbb{C}P^2$, we find $A = (1)$ or $A = (-1)$, depending on the orientation on $\mathbb{C}P^2$.

$\smile$ Non-singular $\Rightarrow \det A = \pm 1$.

$\smile$ Symmetric $\Rightarrow A^t = A$. Picking a different basis for $H^2(M)$ transforms A into $T^t A T$ for $T \in \mathbb{Z}^{m\times m}$ with $\det T = \pm 1$.

Picking the opposite orientation for M transforms A into $-A$.

If $M \simeq N$ via a map $f: M \longrightarrow N$

$$\text{call } f \begin{cases} \text{orientation-preserving (op)} & \text{if } \deg f = 1 \\ \text{-reversing} & \text{if } \deg f = -1 \end{cases}$$

then $A_M = (\deg f) \cdot T^t A_N T$ for some T.

Ex $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ are <u>not</u> o.p. hom. equiv.

since $(1) \neq T^t (-1) T$ for $T = (\pm 1)$.

Thm (Whitehead) The converse holds:

$$M \underset{o.p.}{\simeq} N \text{ iff } A_M = T^t A_N T.$$

(9) Cohomology with compact support & Proof of PD

Proof idea for PD: induction over number of charts, using Mayer-Vietoris to glue charts together. Problem: Union of charts may be non-compact.

Solution: Define a new cohomology theory H^k_c st $H^k_c \cong H^k$ if M compact, and extend PD:

Theorem 1 (PD without compactness assumption) R commutative ring with 1, M^n be oriented. Then we have an isom (to be defined later)

$$PD: H^k_c(M;R) \longrightarrow H_{n-k}(M;R).$$

Motivation for H^k_c X a locally finite Δ-complex, ie every k-simplex is face of only finitely many $(k+1)$-simplexes.

Let the simplicial cochain complex with compact support be

$$C^k_{c\Delta}(X) := \{\varphi \in C^i_\Delta(X) \mid \varphi(\sigma) = 0 \text{ except for finitely many } k\text{-simplexes } \sigma \in X\}$$

Note $C^\bullet_{c\Delta} \subseteq C^\bullet_\Delta$ is a subcomplex.

Eg $X =$

$$\begin{tikzcd} \cdots & \underset{v_{-1}}{\circ} \arrow[r, "e_{-1}"] & \underset{v_0}{\circ} \arrow[r, "e_0"] & \underset{v_1}{\circ} \arrow[r, "e_1"] & \underset{v_2}{\circ} & \cdots \end{tikzcd}$$

$$de_i = e_{i+1} - e_i$$

Since $X \cong \mathbb{R} \Rightarrow H_0^\Delta(X) \cong H^0_\Delta(X) \cong \mathbb{Z}$, $H_k^\Delta(X) \cong H^k_\Delta(X) \cong 0 \quad \forall k \geq 1$,

so PD doesn't hold for $\mathbb{R}$. Let us check that it does when using $H^k_{c\Delta}$!

$$v^i(v_j) = e^i(e_j) = \delta_{ij}.$$

$$C^0_{c\Delta}(X) = \bigoplus_{i\in\mathbb{Z}} v^i \xrightarrow{\quad d^0 \quad} C^1_{c\Delta}(X) = \bigoplus_{i\in\mathbb{Z}} e^i$$

$$v^i \longmapsto e^{i-1} - e^i$$

Since $d^0(v^i)(e_j) = v^i(d_1(e_j)) = v^i(v_{j+1} - v_j) = \delta_{i,j+1} - \delta_{i,j}$

So $\ker d^0 = 0$ and $\operatorname{coker} d^0 \cong \mathbb{Z}$ generated by $[e^i]$ for any i.

$\Rightarrow H^0_\Delta(X) \cong 0$, $H^1_\Delta(X) \cong \mathbb{Z}$ and PD holds.

Def X top. space, A ab. group. Let the singular cochain complex with compact support of X with coefficients in A be

$$C_c^k(X;A) := \{ \varphi \in C^k(X;A) \mid \exists \text{ compact } K \subseteq X \text{ st } \varphi(\sigma) = 0 \text{ for all } \sigma : \Delta^k \to X \text{ with } \mathrm{im}(\sigma) \cap K = \emptyset \}$$

Note $C_c^k \subseteq C^k$ is a subcomplex, because $\varphi \in C_c^k(X;A) \Rightarrow$ $d^k\varphi(\sigma) = \varphi(d_{k+1}\sigma) = 0$ for $\sigma : \Delta^{k+1} \to X$ with $\mathrm{im}(\sigma) \cap K = \emptyset$, since $\mathrm{im}(d\sigma) \subseteq \mathrm{im}(\sigma) \Rightarrow \mathrm{im}(d\sigma) \cap K = \emptyset$.

$H_c^k(X;A) :=$ cohomology of $C_c^\bullet(X;A)$ is called singular cohomology with compact support of X with coefficients in A.

Remark 2 $C_c^k(X;A) = C^k(X;A)$ if X is compact (take $K = X$)

Def Let I be a set partially ordered by $\leq$ (ie $\leq$ is reflexive, antisymmetric and transitive). If $\forall \alpha, \beta \in I \; \exists \gamma \in I$ with $\alpha \leq \gamma$, $\beta \leq \gamma$ then $(I, \leq)$ is called a directed set.

eg subsets of a fixed set X ordered by inclusion, or open subsets of X, or compact subsets of X.