Def Let I be a directed set.

Let A_α be an R-module for each $\alpha \in I$, and $f_{\alpha\beta} : A_\alpha \to A_\beta$ a homom. for each pair $\alpha, \beta \in I$ with $\alpha \leq \beta$, such that

$$f_{\alpha\alpha} = \mathrm{id}_{A_\alpha} \quad \text{and} \quad f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}.$$

A module B with homoms. $g_\alpha : A_\alpha \to B$ for all $\alpha \in I$ st $g_\beta \circ f_{\beta\alpha} = g_\alpha$ $\forall \alpha \leq \beta$ is called direct limit of the A_α, denoted $B = \varinjlim_{\alpha \in I} A_\alpha$, if it satisfies the following universal property: if C is a module with homoms $h_\alpha : A_\alpha \to C$ and $h_\beta \circ f_{\beta\alpha} = h_\alpha$, then $\exists !\ i : B \to C$ st $i \circ g_\alpha = h_\alpha$.

$$\begin{tikzcd}
\cdots \arrow[r] & A_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[d, "g_\alpha"'] \arrow[drr, "h_\alpha" near end] & A_\beta \arrow[r] \arrow[dl, "g_\beta"'] \arrow[dr, "h_\beta"] & \cdots \\
& \varinjlim_{\alpha} A_\alpha \arrow[rr, dashed, "\exists ! i"'] & & C
\end{tikzcd}$$

Prop 2 $\varinjlim A_\alpha$ exists, and is unique up to unique isomorphism.

Pf Existence: $B := \bigoplus_{\alpha \in I} A_\alpha \Big/ \langle x - f_{\alpha\beta}(x) \mid x \in A_\alpha, \alpha \leq \beta \rangle$

and $g_\alpha : A_\alpha \to B$ is composition of $A_\alpha \xhookrightarrow{\text{incl}} \bigoplus A_\alpha \xrightarrow{\text{proj}} B$.

Given C and h_α, let $i : B \to C$ send $[x] \in B$, $x \in A_\alpha$ to $h_\alpha(x)$.

Uniqueness: the usual proof. $\square$

Ex 3 * Every module is the direct limit of its f.g. submodules

* The direct limit of

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "2"] \arrow[dr, "1"'] & \mathbb{Z} \arrow[r, "3"] \arrow[d, "\frac{1}{2}"] & \mathbb{Z} \arrow[r, "4"] \arrow[dl, "\frac{1}{6}"] & \mathbb{Z} \arrow[r, "5"] \arrow[dll, "\frac{1}{4!}"] & \cdots \\
& \mathbb{Q} & & \cdots &
\end{tikzcd}$$

is $\mathbb{Q}$, with maps:

Prop 4 X top. space, $I = \{ K \subset X \mid K \text{ compact} \}$. Then

$$H_c^\ell(X; A) \cong \varinjlim_{K \in I} H^\ell(X, X \setminus K; A).$$

Proof Suppress A coefficients from notation in this proof. Write $L := \varinjlim \cdots$

$$C^\ell(X, X\setminus K) = \{ \varphi \in C^\ell(X) \mid \varphi(\sigma) = 0 \text{ if } \underbrace{\operatorname{im}\sigma \subseteq X \setminus K}_{\iff \operatorname{im}\sigma \cap K = \emptyset} \}$$

So we have an inclusion of cochain complexes

$$j : C^\ell(X, X\setminus K) \longrightarrow C_c^\ell(X)$$

By univ. property, $\exists !\ i : L \longrightarrow H^\ell(X; A)$ st

$$\begin{tikzcd} H^\ell(X, X\setminus K) \arrow[r, "g_K"] \arrow[rr, bend right, "j^*"'] & L \arrow[r, "i"] & H_c^\ell(X) \end{tikzcd}$$ commutes.

<u>i surjective</u> $[\varphi] \in H_c^\ell(X) \Rightarrow \exists$ compact $K \in I$ st $\varphi(\sigma) = 0$ for $\operatorname{im}\sigma \cap K = \emptyset$ $\Rightarrow$ $[\varphi]$ in image of $j^* : H^\ell(X, X\setminus K) \to H_c^\ell(X)$

$\Rightarrow [\varphi] \in \operatorname{im} i$.

<u>i injective</u> $x \in L$ with $i(x) = 0 \Rightarrow$ Pick $K \in I$ st

$x = g_K([\varphi])$ for $[\varphi] \in H^\ell(X, X\setminus K)$. Then $j^*([\varphi]) = i(x) = 0$

$\Rightarrow \exists\ \psi \in C_c^{\ell-1}(X)$ with $d^{\ell-1}\psi = j(\varphi)$. Pick K' compact with

$\psi(\sigma) = 0$ for $\operatorname{im}(\sigma) \cap K' = \emptyset$. Then $\psi \in C^{\ell-1}(X, X \setminus (K \cup K'))$

$\Rightarrow [\varphi] = 0 \in H^\ell(X, X\setminus(K \cup K')) \Rightarrow x = g_{K \cup K'}([\varphi]) = 0$ $\square$

Prop 5 Skipped in lecture

X top. space, $I \subseteq$ Power set of X, partially ordered by inclusion.

Suppose I is directed, $X = \bigcup_{U \in I} U$, and $\forall K \subseteq X$ compact $\exists\, U \in I$ with $K \subseteq U$

(the last property follows eg if all $U \in I$ are open). Then $H_k(U; A)$ with

inclusion-induced maps has direct limit

$$\varinjlim H_k(U; A) \cong H_k(X; A)$$

Proof Exercise, similar to proof of Prop 3. $\square$

Prop 6 $J \subseteq I$ directed sets st $\forall \alpha \in I\ \exists \beta \in J : \alpha \leq \beta$. Then

$$\varinjlim_{\beta \in J} A_\beta \cong \varinjlim_{\alpha \in I} A_\alpha .$$

Proof (Skipped in lecture)

Each A_β has a map $g_\beta : A_\beta \longrightarrow \varinjlim_{\alpha \in I} A_\alpha$. These are compatible with the $f_{\beta\beta'}$. So the universal property for $\varinjlim_{\beta \in J} A_\beta$ yields

$$\varphi : \varinjlim_{\beta \in J} A_\beta \longrightarrow \varinjlim_{\alpha \in I} A_\alpha .$$

Conversely, for each A_α $\exists \tilde{\beta}$ with $\alpha \leq \tilde{\beta}$, and thus a map

$$A_\alpha \xrightarrow{f_{\alpha\tilde{\beta}}} A_{\tilde{\beta}} \xrightarrow{g_{\tilde{\beta}}} \varinjlim_{\beta \in J} A_\beta$$

These are compatible with the $f_{\alpha\alpha'}$. So the univ. property for $\varinjlim_{\alpha \in I} A_\alpha$ yields $\psi : \varinjlim_{\alpha \in I} A_\alpha \longrightarrow \varinjlim_{\beta \in J} A_\beta$.

By the uniqueness parts of the universal properties, $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identities. $\square$

Ex 7 To calculate $H^\bullet_c(\mathbb{R}^n; A)$, use

$$J = \{ B_r(0) \mid r \in \{1, 2, 3, \dots\} \} \subseteq I.$$

We have $H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A) \cong \begin{cases} A & k = n \\ 0 & \text{else} \end{cases}$ by LES of pair.

Inclusions induce isos $H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A) \longrightarrow H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_s(0); A)$ for $r \leq s$. So

$$H^\ell_c(\mathbb{R}^n; A) \cong \varinjlim_{K \in J} H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus K; A) \cong H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A)$$

for any r. Thus $H^\ell_c(\mathbb{R}^n; A) \cong \begin{cases} A & k = n \\ 0 & \text{else} \end{cases}$.

So PD as in Thm 1 holds for $\mathbb{R}^n$!

Theorem 1 with def of map PD Let M^n be R-oriented. Then the map $PD: H^\ell_c(M;R) \longrightarrow H_{n-\ell}(M;R)$ defined as follows is an iso:

For $K \subseteq M$ compact, there is a unique $\mu_K \in H_n(M, M\setminus K; R)$ st μ_K maps to the generator of $H_n(M, M\setminus x; R)$ given by the orientation for all $x \in K$ (Lemma 7.3). The relative cap product yields a map

$$H^\ell(M, M\setminus K; R) \xrightarrow{h_K} H_{n-\ell}(M;R),$$

$$[\varphi] \longmapsto \mu_K \frown [\varphi]$$

If $L \subseteq M$ compact, $K \subseteq L$, then $incl_*(\mu_L) = \mu_K$. Using that and naturality of relative cap product, the following commutes:

$$\begin{tikzcd}
H^\ell(M, M\setminus K; R) \arrow[r, "h_K"] \arrow[d, "incl^*"] & H_{n-\ell}(M;R) \\
H^\ell(M, M\setminus L; R) \arrow[ur, "h_L"'] &
\end{tikzcd}$$

So the univ. property yields a map

$$\varinjlim_K H^\ell(M, M\setminus K; R) \longrightarrow H_{n-\ell}(M;R).$$

Precomposing with the isom. $H^\ell_c(M;R) \longrightarrow \varinjlim_K H^\ell(M, M\setminus K;R)$ gives our map PD!