

**Def** Let  $I$  be a directed set.

Let  $A_\alpha$  be an  $R$ -module for each  $\alpha \in I$ , and  $f_{\alpha\beta}: A_\alpha \rightarrow A_\beta$  a homom. for each pair  $\alpha, \beta \in I$  with  $\alpha \leq \beta$ , such that

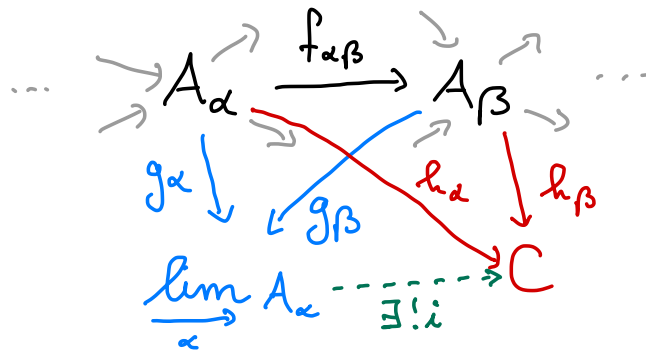
$$f_{\alpha\alpha} = \text{id}_{A_\alpha} \quad \text{and} \quad f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}.$$

A module  $B$  with homom.  $g_\alpha: A_\alpha \rightarrow B$  for all  $\alpha \in I$  st

$g_\beta \circ f_{\beta\alpha} = g_\alpha \quad \forall \alpha \leq \beta$  is called **direct limit of the  $A_\alpha$** ,

denoted  $B = \varinjlim_{\alpha \in I} A_\alpha$ , if it satisfies the following universal property:

if  $C$  is a module with homom.  $h_\alpha: A_\alpha \rightarrow C$  and  $h_\beta \circ f_{\beta\alpha} = h_\alpha$ , then  $\exists!$   $i: B \rightarrow C$  st  $i \circ g_\alpha = h_\alpha$ .



**Prop 2**  $\varinjlim A_\alpha$  exists, and is unique up to unique isomorphism.

**PS** Existence:  $B := \left( \bigoplus_{\alpha \in I} A_\alpha \right) / \langle x - f_{\alpha\beta}(x) \mid x \in A_\alpha, \alpha \leq \beta \rangle$

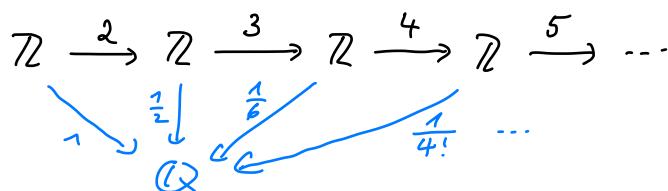
and  $g_\alpha: A_\alpha \rightarrow B$  is composition of  $A_\alpha \xrightarrow{\text{incl}} \bigoplus A_\alpha \xrightarrow{\text{proj}} B$ .

Given  $C$  and  $h_\alpha$ , let  $i: B \rightarrow C$  send  $[x] \in B$ ,  $x \in A_\alpha$  to  $h_\alpha(x)$ .

Uniqueness: the usual proof.  $\square$

**Ex 3** \* Every module is the direct limit of its f.g. submodules

\* The direct limit of



is  $\mathbb{Q}$ , with maps:

**Prop 4**  $X$  top. space,  $I = \{K \subset X \mid K \text{ compact}\}$ . Then

$$H_c^l(X; A) \cong \varinjlim_{K \in I} H^l(X, X \setminus K; A).$$

**Proof** Suppress  $A$  coefficients from notation in this proof. Write  $L := \varinjlim \dots$

$$C^l(X, X \setminus K) = \{ \varphi \in C^l(X) \mid \varphi(\sigma) = 0 \text{ if } \underbrace{\text{im } \sigma \subseteq X \setminus K} \}$$

$\Leftrightarrow \text{im } \sigma \cap K = \emptyset$

So we have an inclusion of cochain complexes

$$j: C^l(X, X \setminus K) \longrightarrow C_c^l(X)$$

By univ. property,  $\exists!$   $i: L \rightarrow H^l(X; A)$  st

$$H^l(X, X \setminus K) \xrightarrow{g_K} L \xrightarrow{i} H_c^l(X) \text{ commutes.}$$

$\underbrace{\hspace{10em}}_{j^*}$

$i$  surjective  $[\varphi] \in H_c^l(X) \Rightarrow \exists$  compact  $K \in I$  st  $\varphi(\sigma) = 0$

for  $\text{im } \sigma \cap K = \emptyset \Rightarrow [\varphi]$  in image of  $j^*: H^l(X, X \setminus K) \rightarrow H_c^l(X)$

$\Rightarrow [\varphi] \in \text{im } i$ .

$i$  injective  $x \in L$  with  $i(x) = 0 \Rightarrow$  Pick  $K \in I$  st

$x = g_K([\varphi])$  for  $[\varphi] \in H^l(X, X \setminus K)$ . Then  $j^*([\varphi]) = i(x) = 0$

$\Rightarrow \exists \psi \in C_c^{l-1}(X)$  with  $d^{l-1}\psi = j([\varphi])$ . Pick  $K'$  compact with

$\psi(\sigma) = 0$  for  $\text{im } (\sigma) \cap K' = \emptyset$ . Then  $\psi \in C^{l-1}(X, X \setminus (K \cup K'))$

$\Rightarrow [\varphi] = 0 \in H^l(X, X \setminus (K \cup K')) \Rightarrow x = g_{K \cup K'}([\varphi]) = 0 \quad \square$

**Prop 5** Skipped in lecture

$X$  top. space,  $I \subseteq$  Power set of  $X$ , partially ordered by inclusion.

Suppose  $I$  is directed,  $X = \bigcup_{U \in I} U$ , and  $\forall K \subseteq X$  compact  $\exists U \in I$  with  $K \subseteq U$

(the last property follows eg if all  $U \in I$  are open). Then  $H_k(U; A)$  with

inclusion-induced maps has direct limit

$$\varinjlim H_k(U; A) \cong H_k(X; A)$$

**Proof** Exercise, similar to proof of Prop 3.  $\square$

Prop 6  $J \subseteq I$  directed sets st  $\forall \alpha \in I \exists \beta \in J : \alpha \leq \beta$ . Then

$$\varinjlim_{\beta \in J} A_\beta \cong \varinjlim_{\alpha \in I} A_\alpha.$$

Proof (Skipped in lecture)

Each  $A_\beta$  has a map  $g_\beta: A_\beta \rightarrow \varinjlim_{\alpha \in I} A_\alpha$ . These are compatible with the  $f_{\beta\beta'}$ . So the universal property for  $\varinjlim_{\beta \in J} A_\beta$  yields

$$\varphi: \varinjlim_{\beta \in J} A_\beta \rightarrow \varinjlim_{\alpha \in I} A_\alpha.$$

Conversely, for each  $A_\alpha \exists \tilde{\beta}$  with  $\alpha \leq \tilde{\beta}$ , and thus a map

$$A_\alpha \xrightarrow{f_{\alpha\tilde{\beta}}} A_{\tilde{\beta}} \xrightarrow{g_{\tilde{\beta}}} \varinjlim_{\beta \in J} A_\beta$$

These are compatible with the  $f_{\alpha\alpha'}$ . So the univ. property for  $\varinjlim_{\alpha \in I} A_\alpha$  yields  $\psi: \varinjlim_{\alpha \in I} A_\alpha \rightarrow \varinjlim_{\beta \in J} A_\beta$ .

By the uniqueness parts of the universal properties,  $\varphi \circ \psi$  and  $\psi \circ \varphi$  are the identities. □

Ex 7 To calculate  $H_c^k(\mathbb{R}^n; A)$ , use

$$J = \{ B_r(0) \mid r \in \{1, 2, 3, \dots\} \} \subseteq I.$$

We have  $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A) \cong \begin{cases} A & k=n \\ 0 & \text{else} \end{cases}$  by LES of pair.

Inclusions induce isos  $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A) \rightarrow H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_s(0); A)$  for  $r \leq s$ . So

$$H_c^k(\mathbb{R}^n; A) \cong \varinjlim_{k \in J} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K; A) \cong H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r(0); A)$$

for any  $r$ . Thus  $H_c^k(\mathbb{R}^n; A) \cong \begin{cases} A & k=n \\ 0 & \text{else} \end{cases}$ .

So PD as in Thm 1 holds for  $\mathbb{R}^n$ !

Theorem 1 with def of map PD Let  $M^m$  be  $\mathbb{R}$ -oriented. Then the map

$$PD: H_c^l(M; \mathbb{R}) \longrightarrow H_{m-l}(M; \mathbb{R})$$
 defined as follows is an iso:

For  $K \subseteq M$  compact, there is a unique  $\mu_K \in H_m(M, M \setminus K; \mathbb{R})$  st  $\mu_K$  maps to the generator of  $H_m(M, M \setminus \infty; \mathbb{R})$  given by the orientation for all  $x \in K$  (Lemma 7.3). The relative cap product yields a map

$$H^l(M, M \setminus K; \mathbb{R}) \xrightarrow{h_K} H_{m-l}(M; \mathbb{R}),$$

$$[\varphi] \longmapsto \mu_K \frown [\varphi]$$

If  $L \subseteq M$  compact,  $K \subseteq L$ , then  $\text{incl}_*(\mu_L) = \mu_K$ . Using that and naturality of relative cap product, the following commutes:

$$\begin{array}{ccc}
 H^l(M, M \setminus K; \mathbb{R}) & \xrightarrow{h_K} & H_{m-l}(M; \mathbb{R}) \\
 \downarrow \text{incl}^* & & \nearrow h_L \\
 H^l(M, M \setminus L; \mathbb{R}) & & 
 \end{array}$$

So the univ. property yields a map

$$\varinjlim_K H^l(M, M \setminus K; \mathbb{R}) \longrightarrow H_{m-l}(M; \mathbb{R}).$$

Precomposing with the isom.  $H_c^l(M; \mathbb{R}) \xrightarrow{\cong} \varinjlim_K H^l(M, M \setminus K; \mathbb{R})$  gives our map PD!