Remark 8 X Hausdorff, $U \subset X$ open, $K \subset U$ compact.

Excise $X \setminus U \implies H^\ell(X, X\setminus K; A) \xrightarrow{\text{incl}^*} H^\ell(U, U\setminus K; A)$

(using that X Hausdorff $\implies K$ is closed $\implies X\setminus K$ open,

So $\quad X\setminus U = \overline{X\setminus U} \subseteq (X\setminus K)^\circ = X\setminus K$)

is an iso. Its inverse composed with g_K is a map

$$H^\ell(U, U\setminus K; A) \longrightarrow H^\ell_c(X; A)$$

By univ. property, these maps induce

$$H^\ell_c(U; A) \longrightarrow H^\ell_c(X; A)$$

So H^ℓ_c is covariantly (!) functorial with respect to inclusions of open subsets of a Hausdorff space.

Lemma 9 M^n R-oriented, $U, V \subseteq M^n$ open, $M = U \cup V$.

Then the following diagram has exact rows and commutes up to sign (R coefficients suppressed from notation):

see Rmk 8

$$\begin{tikzcd}
\cdots \arrow[r] & H^\ell_c(U\cap V) \arrow[r, "\begin{pmatrix} \text{incl}_* \\ -\text{incl}_* \end{pmatrix}"] \arrow[d, "PD_{U\cap V}"'] & H^\ell_c(U) \oplus H^\ell_c(V) \arrow[r, "(\text{incl}_* \ \ \text{incl}_*)"] \arrow[d, "\begin{pmatrix} PD_U & 0 \\ 0 & PD_V \end{pmatrix}"] & H^\ell_c(M) \arrow[r, "\text{to be defined}"] \arrow[d, "PD_M"] & H^{\ell+1}_c(U\cap V) \arrow[r] \arrow[d, "PD_{U\cap V}"] & {} \\
\cdots \arrow[r] & H_{n-\ell}(U\cap V) \arrow[r, "\begin{pmatrix} \text{incl}_* \\ -\text{incl}_* \end{pmatrix}"'] & H_{n-\ell}(U) \oplus H_{n-\ell}(V) \arrow[r, "(\text{incl}_* \ \ \text{incl}_*)"'] & H_{n-\ell}(M) \arrow[r, "\partial"'] & H_{n-\ell-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Proof Let $K \subseteq U$, $L \subseteq V$ be compact. Consider the following diagram.

Top Row: Relative MV exact seq.

Bottom Row: MV exact seq

Middle Row: Maps chosen so that first two rows commute (possible since vertical maps are isos by excision) $\Rightarrow$ is exact.

$$
\begin{tikzcd}
\cdots \arrow[r] & H^\ell(M, M\setminus(K\cap L)) \arrow[r, "\begin{pmatrix} incl^* \\ -incl^* \end{pmatrix}"] \arrow[d, "\cong"', "incl^*"] & H^\ell(M, M\setminus K) \oplus H^\ell(M, M\setminus L) \arrow[r, "\begin{pmatrix} incl^* & incl^* \end{pmatrix}"] \arrow[d, "\cong"', "\begin{pmatrix} incl^* & 0 \\ 0 & incl^* \end{pmatrix}"] & H^\ell(M, M\setminus(K\cup L)) \arrow[r, "\delta"] \arrow[d, "\cong"', "id"] & H^{\ell+1}(M, M\setminus(K\cap L)) \arrow[r] \arrow[d, "\cong"', "incl^*"] & \cdots \\
\cdots \arrow[r] & H^\ell(U\cap V, (U\cap V)\setminus(K\cap L)) \arrow[r] \arrow[d, "\mu_{K\cap L}\frown"] \arrow[dr, phantom, "A"] & H^\ell(U, U\setminus K) \oplus H^\ell(V, V\setminus L) \arrow[r] \arrow[d, "\begin{pmatrix} \mu_K\frown & 0 \\ 0 & \mu_L\frown \end{pmatrix}"] \arrow[dr, phantom, "B"] & H^\ell(M, M\setminus(K\cup L)) \arrow[r] \arrow[d, "\mu_{K\cup L}\frown"] \arrow[dr, phantom, "C"] & H^{\ell+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \arrow[r] \arrow[d, "\mu_{K\cap L}\frown"] & \cdots \\
\cdots \arrow[r] & H_{n-\ell}(U\cap V) \arrow[r, "\begin{pmatrix} incl_* \\ -incl_* \end{pmatrix}"'] & H_{n-\ell}(U) \oplus H_{n-\ell}(V) \arrow[r, "\begin{pmatrix} incl_* & incl_* \end{pmatrix}"'] & H_{n-\ell}(M) \arrow[r, "\partial"'] & H_{n-\ell-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}
$$

Steps (1) Commutativity squares A, B

(2) —— " —— C

(3) Taking direct limit

(3) Assume A, B, C commute. Let $I = \{(K, L) \mid K \subseteq U \text{ compact}, L \subseteq V \text{ compact}\}$. Taking $\varinjlim_{(K,L)\in I}$ of the middle row gives the top row of the statement of the lemma, using that every compact subset of M can be written as $K \cup L$ for $K \subseteq U$, $L \subseteq V$ (exercise). Moreover, $\varinjlim$ preserves exactness (exercise). And $\varinjlim$ of the lower vertical maps are the PD maps in the lemma.

So the lemma follows once commutativity of A, B, C has been established in Steps (1), (2).

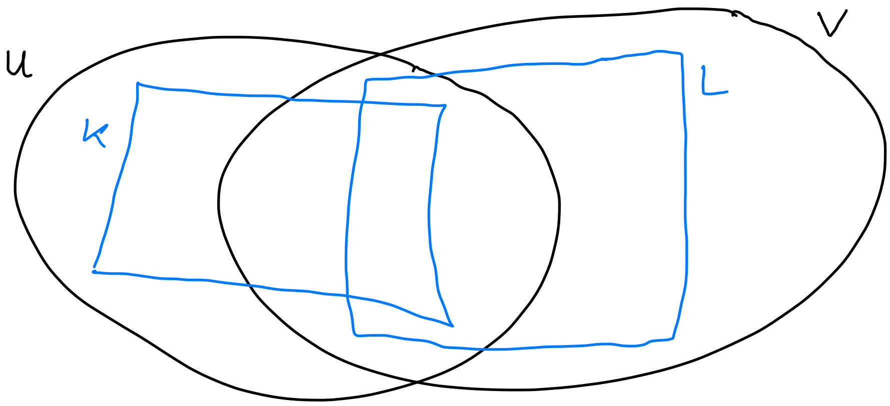
(1) Commutativity of Ⓐ, Ⓑ follows from naturality of the relative cap product, and $incl_*(\mu_K) = \mu_{K\cap L}$.

(2) To show: Commutativity of

$$\begin{tikzcd}
H^{\ell}(M, M\setminus(K\cup L)) \arrow[r, "\delta"] \arrow[dd, "\mu_{K\cup L}\frown"'] \arrow[ddr, phantom, "C"] & H^{\ell+1}(M, M\setminus(K\cap L)) \arrow[d, "\cong"', "incl^*"] \\
& H^{\ell+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \arrow[d, "\mu_{K\cap L}\frown"] \\
H_{n-\ell}(M) \arrow[r, "\partial"'] & H_{n-\ell-1}(U\cap V)
\end{tikzcd}$$

dropping incl from notation

By barycentric subdivision, $\mu_{K\cup L} = [\underbrace{\alpha_{U\setminus L}}_{\in C_n(U\setminus L)} + \underbrace{\alpha_{U\cap V}}_{\in C_n(U\cap V)} + \underbrace{\alpha_{V\setminus K}}_{\in C_n(V\setminus K)}]$



$\Rightarrow \mu_{K\cap L} = incl_*(\mu_{K\cup L}) = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$

$= [\alpha_{U\cap V}] \in H_n(M, M\setminus(K\cap L))$

Since $U\setminus L \subseteq M\setminus(K\cap L)$, so $\alpha_{U\setminus L} = 0 \in C_n(M, M\setminus(K\cap L))$,
and similarly for $\alpha_{V\setminus K}$

Similarly, $\mu_K = incl_*(\mu_{K\cup L}) = [\alpha_{U\setminus L} + \alpha_{U\cap V}]$.

Let $[\varphi] \in H^\ell(M, M \setminus (K \cup L))$. We need to check that in ©,

$$\mu_{K\cap L} \frown \mathrm{incl}^*(\delta([\varphi])) = \partial(\mu_{K\cup L} \frown [\varphi]).$$

Clockwise image of $[\varphi]$

How to calculate $\delta([\varphi])$?

Write $\varphi = \psi_K - \psi_L$ st $\psi_K \in C^\ell(M, M \setminus K)$, $\psi_L \in C^\ell(M, M \setminus L)$.
Then $\delta[\varphi] = [d^\ell \psi_K] = [d^\ell \psi_L]$ (relative cohom. MV connecting hom.)

$$\text{So } \mu_{K\cap L} \frown \mathrm{incl}^*(\delta([\varphi])) = [\alpha_{U\cap V} \frown d^\ell \psi_K]$$
$$= \boxed{[d_n(\alpha_{U\cap V}) \frown \psi_K]}$$

since $\underbrace{d_{m-\ell}(\alpha_{U\cap V} \frown \psi_K)}_{= 0 \in H_{m-\ell-1}(U\cap V)} = (-1)^\ell (d_m \alpha_{U\cap V} \frown \psi_K - \alpha_{U\cap V} \frown d^\ell \psi_K)$

Counterclockwise image of $[\varphi]$

$$\partial(\mu_{K\cup L} \frown [\varphi]) = \partial\Big(\big[\underbrace{\alpha_{U\setminus L} \frown \varphi}_{\text{chain in } U} + \overbrace{\alpha_{U\cap V} \frown \varphi + \alpha_{V\setminus K} \frown \varphi}^{\text{chain in } V}\big]\Big)$$

$= [d_{m-\ell}(\alpha_{U\setminus L} \frown \varphi)]$ by def of MV connecting homom. ∂ (cracking the egg)

$= (-1)^\ell [d_m(\alpha_{U\setminus L}) \frown \varphi]$ since $d^\ell \varphi = 0$

$= (-1)^\ell [d_m(\alpha_{U\setminus L}) \frown \psi_K]$ since $d_m(\alpha_{U\setminus L}) \frown \psi_L = 0$

$$= \boxed{(-1)^{\ell+1} [d_m(\alpha_{U\cap V}) \frown \psi_K]}$$

since: $[\alpha_{U\setminus L} + \alpha_{U\cap V}] = \mu_K \Rightarrow d_m(\alpha_{U\setminus L} + \alpha_{U\cap V}) \in C_{m-1}(M \setminus K)$

$\Rightarrow d_m(\alpha_{U\setminus L} + \alpha_{U\cap V}) \frown \psi_K = 0.$ □