Previously

Theorem 1 M^n R-oriented $\Rightarrow H^\ell_c(M;R) \xrightarrow{PD_M} H_{n-\ell}(M;R)$ is an iso.

Lemma 9 M^n R-oriented, $U,V \subseteq M^n$ open, $M = U\cup V$.

$$\begin{tikzcd}
\to H^\ell_c(U\cap V) \arrow[r] \arrow[d, "PD_{U\cap V}"'] & H^\ell_c(U)\oplus H^\ell_c(V) \arrow[r] \arrow[d, "\begin{pmatrix} PD_U & 0 \\ 0 & PD_V \end{pmatrix}"] & H^\ell_c(M) \arrow[r] \arrow[d, "PD_M"] & H^{\ell+1}_c(U\cap V) \to \arrow[d, "PD_{U\cap V}"] \\
\to H_{n-\ell}(U\cap V) \arrow[r] & H_{n-\ell}(U)\oplus H_{n-\ell}(V) \arrow[r] & H_{n-\ell}(M) \arrow[r] & H_{n-\ell-1}(U\cap V) \to
\end{tikzcd}$$

has exact rows & commutes up to sign.

Now **Fixing the sign in Lemma 9**

In all squares that anticommute (ie $\downarrow\rightarrow = -\rightarrow\downarrow$), simply switch the sign of one of the horizontal maps. This preserves exactness and yields commutativity.

Proof of Theorem 1

(A) If $M = U\cup V$ for U,V open and PD_U, PD_V, $PD_{U\cap V}$ are isos, then so is PD_M. Proof: Five-lemma & Lemma 9 ✓

(B) If $M = \bigcup_{i=1}^\infty U_i$ with $U_1 \subseteq U_2 \subseteq \cdots$ open, and all PD_{U_i} are isos, then so is PD_M. Proof: Rmk 8 yields a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^\ell_c(U_i) \arrow[r, "incl_*"] \arrow[dr, "incl_*"'] & H^\ell_c(U_{i+1}) \arrow[r] \arrow[d, "incl_*"] & \cdots \\
& & H^\ell_c(M) &
\end{tikzcd}$$

The induced map $\varinjlim_i H^\ell_c(U_i) \to H^\ell_c(M)$ is an iso, since $K\subseteq M$ compact $\Rightarrow \exists i: K\subseteq U_i$.

Moreover, $\varinjlim_i H_{n-\ell}(U_i) \cong H_{n-\ell}(M)$ (Prop 5 / Ex. on Sheet 6).

By assumption, all $PD_{U_i}: H^\ell_c(U_i) \to H_{n-\ell}(U_i)$ are isos.

So the induced map $\varinjlim_i H^\ell_c(U_i) \longrightarrow \varinjlim_i H_{n-\ell}(U_i)$ is also an iso.

It equals PD_M.

(1) $\underline{M = \mathbb{R}^n}$ We already know $H^\ell_c(\mathbb{R}^n) = \mathbb{R}^{\delta_{\ell,n}} \cong H_{n-\ell}(\mathbb{R}^n)$ (Ex 7), but still need to check that PD_M is an iso.

Let $f: (\mathbb{R}^n, \mathbb{R}^n \setminus B_1(0)) \to (\Delta^n, \partial\Delta^n)$ be a hom. equiv. Then the following commutes (left triangle by def of PD_M, right square by naturality of rel. cap product):

$$\begin{tikzcd}
H^n_c(\mathbb{R}^n) \arrow[dr, bend right, "PD_M"'] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_1(0)) \arrow[l, "\text{iso by Ex 7}"'] \arrow[d, "\mu_{B_1(0)} \frown"] & H^n(\Delta^n, \partial\Delta^n) \arrow[l, "f^*"', "\cong"] \arrow[d, "f_*(\mu_{B_1(0)}) \frown"] \\
 & H_0(\mathbb{R}^n) \arrow[r, "\cong", "f_*"'] & H_0(\Delta^n)
\end{tikzcd}$$

So it suffices to check that $f_*(\mu_{B_1(0)}) \frown$ is an iso, which can be seen using simplicial (co-) homology. ✓

(2) $\underline{M \subseteq \mathbb{R}^n,\quad M = V_1 \cup \dots \cup V_k \text{ for } V_i \text{ open and convex}}$

By induction over k. For $k=1$, follows from (1) since $V_i \cong \mathbb{R}^n$. If true up to k:

$M = U \cup V_{k+1}$ for $U = V_1 \cup \dots \cup V_k$. $PD_{V_{k+1}}$ iso by (1), PD_U iso by induction hypothesis, and $PD_{U \cap V_{k+1}}$ iso also by induction hypothesis, since
$$U \cap V_{k+1} = (U \cap V_1) \cup \dots \cup (U \cap V_k)$$
with $U \cap V_i$ open and convex. So PD_M iso by (A). ✓

(3) $\underline{M \subseteq \mathbb{R}^n \text{ open}}$ Write $M = \bigcup_{i=1}^\infty V_i$ with V_i open and convex.

(eg take as U_i all open balls $\subseteq M$ with rational radius and rational coordinates)

Let $U_k = V_1 \cup \dots \cup V_k$. Then PD_{U_k} iso for all k by (2). Done by (B). ✓

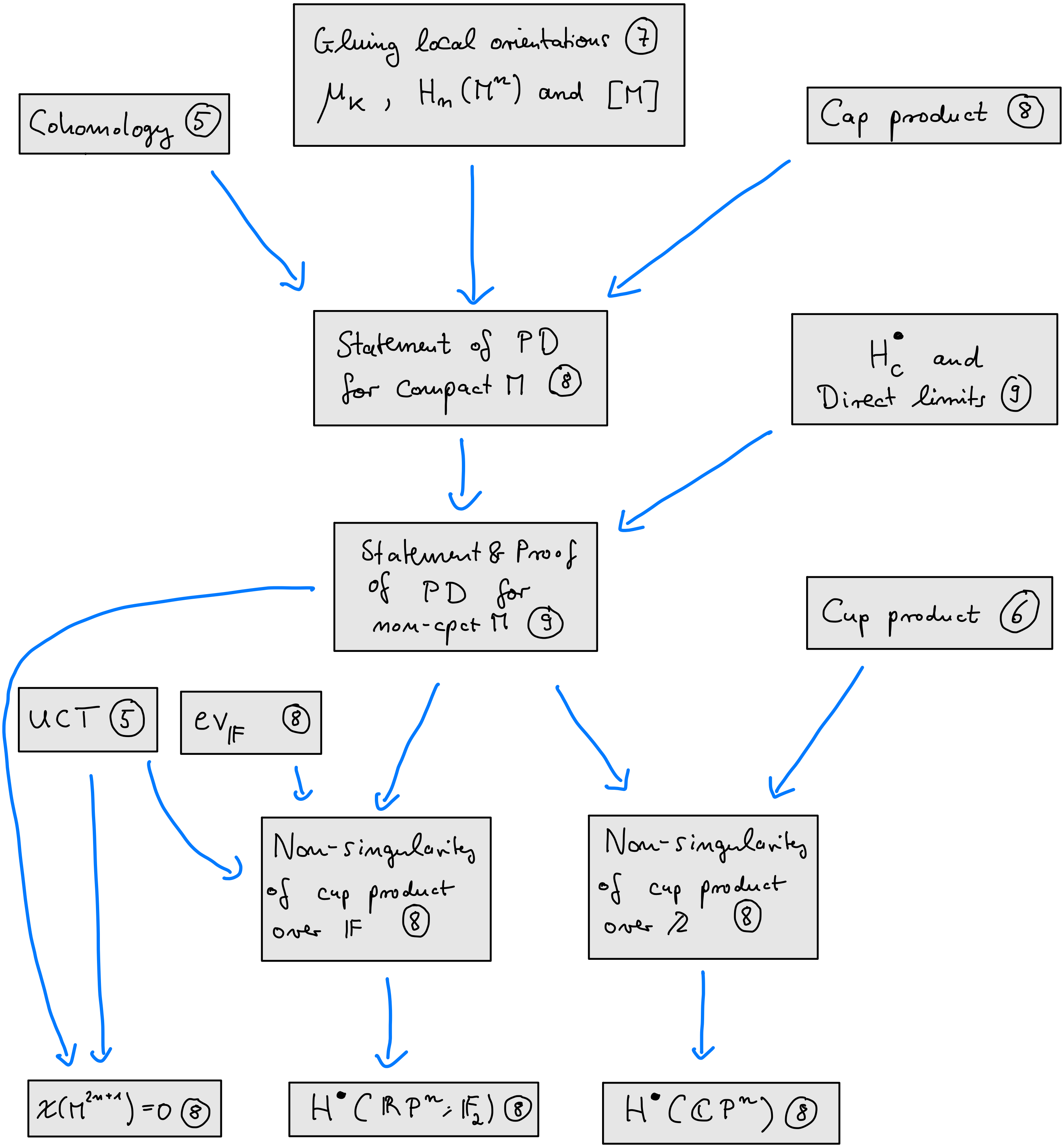
(4) $\underline{M \text{ with finite atlas}}$, ie $M = V_1 \cup \dots \cup V_k$ with V_i open and $\cong \mathbb{R}^n$.

Proceed as in (2), using (3) on $U \cap V_{k+1}$, which is homeo to an open set $\subseteq \mathbb{R}^n$

(5) $\underline{\text{General } M}$ has a countable atlas (using 2nd countability), ie

$M = \bigcup_{i=1}^\infty V_i$ with V_i open and $\cong \mathbb{R}^n$. Proceed as in (4). □

Logical structure of Ch. 5–9 (ie Cohomology)



(10) Alexander Duality

Theorem 1 (Alexander Duality) Let $n \geq 0$ and $K \subseteq S^n$ be a locally contractible, compact subspace, $K \neq \emptyset$, $K \neq S^n$.

Then for all i

$$\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

Remark 2 This means: if a compact top. space K is locally "tame" (ie locally contractible, eg a manifold), and you embed K into a sphere S^n, then the homology of the complement $S^n \setminus K$ does not depend on the choice of embedding!