Example 3 $K \subseteq S^3$ with $K \cong S^1$ is called a knot. By Alexander Duality, $\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{2-i}(S^1; \mathbb{Z})$

$H_0(S^n \setminus K) \cong H_1(S^n \setminus K) \cong \mathbb{Z}$, and all other homology groups are trivial. This is easy to see geometrically for an "unknotted" K, since then $S^3 \setminus K \simeq S^1$.

Corollary 4 M^{n-1} non-orientable, compact. Then M does not embed into S^n.

Proof Assume $M \neq \emptyset$, $M \subseteq S^n$. By Alexander duality:

$$H^{n-1}(M) \cong \tilde{H}^{n-1}(M) \cong \tilde{H}_0(S^n \setminus M)$$

So $H^{n-1}(M)$ is free. By UCT we also know:

$$H^{n-1}(M) \cong \mathrm{Hom}(\underbrace{H_{n-1}(M)}_{\substack{\cong 0 \text{ because} \\ M \text{ non-orientable} \\ (\text{Prop } 7.2\ (ii))}}, \mathbb{Z}) \oplus \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(M), \mathbb{Z})}_{\substack{\cong T(H_{n-2}(M)) \\ \text{since } H_{n-2}(M) \text{ f.g.}}}$$

$H^{n-1}(M)$ free $\Rightarrow$ Ext-term zero $\Rightarrow$ $H_{n-2}(M)$ free and $H^{n-1}(M) \cong 0$.

Again by UCT:

$$H^{n-1}(M; \mathbb{F}_2) \cong \underbrace{\mathrm{Hom}(H_{n-1}(M), \mathbb{F}_2)}_{\cong 0 \text{ as above}} \oplus \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(M), \mathbb{F}_2)}_{0 \text{ because } H_{n-2}(M) \text{ free}}$$

$\Rightarrow H^{n-1}(M; \mathbb{F}_2) \cong 0$. But PD $\Rightarrow$ $H^{n-1}(M; \mathbb{F}_2) \cong H_0(M; \mathbb{F}_2)$, which is non-trivial since $M \neq \emptyset$. Contradiction. $\square$

Lemma 4 $K \subseteq \mathbb{R}^n$ with K compact and locally contractible.

(1) There is $U_0 \subseteq \mathbb{R}^n$ open with $K \subseteq U_0$ and a retraction $r: U_0 \to K$.

(2) For all open $U \subseteq U_0$ with $K \subseteq U$, there exists an open $V \subseteq U$ with $K \subseteq V$ st $\mathrm{incl}_{V \hookrightarrow U}$ is homotopic to $\mathrm{incl}_{K \hookrightarrow U} \circ r|_V$.

Proof (1) Hatcher Thm A.7

(2) Skipped in Lecture

Because we're in $\mathbb{R}^n$, one may simply define a "linear" homotopy

$$h: U \times I \to \mathbb{R}^n, \quad h(x,t) = (1-t)x + t\,r(x)$$

between id_U and r. However, this is a homotopy through maps to $\mathbb{R}^n$, not maps to U. $h^{-1}(U)$ is open in $U \times I$. By def. of the product topology, for every $t \in I$ there is $V_t \subseteq U$ open, $\varepsilon_t > 0$ such that

$$V_t \times ((t-\varepsilon_t, t+\varepsilon_t) \cap [0,1]) \subseteq h^{-1}(U).$$

We have $[0,1] = \bigcup_{t \in [0,1]} (t-\varepsilon_t, t+\varepsilon_t) \cap [0,1]$, and since $[0,1]$ is compact, there is a finite subcovering. The intersection of the corresponding V_t is an open set V such that $V \times I \subseteq h^{-1}(U) \implies$ h yields a homotopy from $V \hookrightarrow U$ to $r|_V$ through maps to U. $\square$

Proof of Theorem 1 Treat the case $i \neq 0$ first. Then

$$
\begin{aligned}
\tilde{H}_i(S^n\setminus K) &\cong H_i(S^n\setminus K) \\
&\cong H_c^{n-i}(S^n\setminus K) && \color{blue}{PD^{-1}} \\
&\cong \varinjlim_{\substack{L\subseteq S^n\setminus K \\ \text{kompakt}}} H^{n-i}(S^n\setminus K,\ S^n\setminus(K\cup L)) && \color{blue}{\text{by Prop 9.4}} \\
&\cong \varinjlim_{L} H^{n-i}(S^n,\ S^n\setminus L) && \color{blue}{\text{incl}^* \text{ is iso by excision}} \\
&\cong \varinjlim_{L} \tilde{H}^{n-i-1}(S^n\setminus L) && \color{blue}{\text{LES of pair}} \\
&\cong \tilde{H}^{n-i-1}(K)
\end{aligned}
$$

<u>Proof of the last iso:</u> Let us prove iso for unreduced cohomology. This implies iso for reduced. Pick $p\in S^n\setminus K$. Then $K\subseteq S^n\setminus p \cong \mathbb{R}^n$. So one may pick U_0 as in Lemma 4 (1) and retraction $r: U_0 \to K$.

By Prop 9.6, $\varinjlim_{L} \cong \varinjlim_{S^n\setminus U_0\subseteq L}$. Then, the universal property yields a map s:

$$
\begin{tikzcd}
H^{n-i-1}(K) \arrow[r, bend right, "(r|_{S^n\setminus L})^*"'] \arrow[dr, bend right, "(r|_{S^n\setminus L'})^*"'] & H^{n-i-1}(S^n\setminus L) \arrow[l, "\mathrm{incl}^*"'] \arrow[r, "g_L"] \arrow[d, "f_{L,L'}=\mathrm{incl}^*"] & \varinjlim_{S^n\setminus U_0\subseteq L\subseteq S^n\setminus K} H^{n-i-1}(S^n\setminus L) \arrow[ll, bend right, "s"'] \\
& H^{n-i-1}(S^n\setminus L') \arrow[ur, "g_L'"'] &
\end{tikzcd}
$$

Let us show that s is an iso.

Surjectivity of s: $\quad \tau|_{S^n \setminus L} \circ \mathrm{incl} = \mathrm{id}_K \Rightarrow$

$\mathrm{incl}^* \circ (\tau|_{S^n\setminus L})^* = \mathrm{id}_{H^{n-i-1}(K)} \Rightarrow \mathrm{incl}^*$ surjective. ✓

Injectivity of s:

Let $x \in \varinjlim$ with $s(x) = 0$ be given. Pick L such that $x = g_L(y)$

$\Rightarrow s(x) = \mathrm{incl}^*(y) = 0$. By Lemma 4(2), pick L' with $L \subseteq L' \subseteq K$

st $S^n \setminus L' \hookrightarrow S^n \setminus L$ is homotopic to $\tau|_{S^n\setminus L'}$.

$\Rightarrow f_{L,L'} = (\tau|_{S^n\setminus L'})^* \circ \mathrm{incl}^* \Rightarrow f_{L,L'}(y) = (\tau|_{S^n\setminus L'})^*(\mathrm{incl}^*(y)) = 0$

$\Rightarrow x = g_L(y) = g_{L'}(f_{L,L'}(y)) = 0.$ ✓

Case $i=0$: As before, we have $H_0(S^n\setminus K) \cong \varinjlim_L H^n(S^n, S^n\setminus L)$.

LES of pair:

$$\underbrace{\tilde H^{n-1}(S^n)}_{\cong 0} \to H^{n-1}(S^n\setminus L) \to H^n(S^n, S^n\setminus L) \to \underbrace{H^n(S^n)}_{\cong \mathbb{Z}} \to \overbrace{H^n(S^n\setminus L)}^{\substack{\cong 0 \text{ since } L\neq\emptyset,\, L\neq S^n \\ \Rightarrow S^n\setminus L \text{ non-compact manifold}}}$$

$\Rightarrow H^n(S^n, S^n\setminus L) \cong H^{n-1}(S^n\setminus L) \oplus \mathbb{Z}.$

$\Rightarrow H_0(S^n\setminus K) \cong H^{n-1}(K) \oplus \mathbb{Z}$

$\Rightarrow \tilde H_0(S^n\setminus K) \cong H^{n-1}(K).$ $\square$