(11) Künneth Theorem (not in exam)

If A and B are R-algebras, then $A \otimes_R B$ is too, via

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'.$$

If A and B are graded, then $A \otimes_R B$ is too, via

$$\deg(a \otimes b) = \deg a + \deg b.$$

Theorem 1 X, Y spaces, $H^i(X;R)$ free of finite rank for all i.

Then there is an isomorphism of graded R-modules

$$H^\bullet(X \times Y; R) \cong H^\bullet(X;R) \otimes_R H^\bullet(Y;R),$$

The ring structure is respected up to sign

Example 2 $H^\bullet(S^1 \times S^1) \cong H^\bullet(S^1) \otimes_{\mathbb{Z}} H^\bullet(S^1)$

as graded rings, with multiplication only respected up to sign.

$$\cong \mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[y]/(y^2)$$

$$\cong \mathbb{Z}[x,y]/(x^2,y^2)$$

This is the result we obtained in Example 6.8, up to sign

Example 3 $H^\bullet(S^2 \times S^4) \cong \mathbb{Z}[x,y]/(x^2,y^2)$

with $\deg x = 2$, $\deg y = 4$. This iso respects multiplication including the sign, since all degrees are even. More explicitly:

$$H^2(S^2 \times S^4) \cong \mathbb{Z} \text{ gen. by } x \qquad x^2 = 0$$

$$H^4(S^5 \times S^4) \cong \mathbb{Z} \text{ gen. by } y \qquad y^2 = 0$$

$$H^6(S^2 \times S^4) \cong \mathbb{Z} \text{ gen. by } xy$$

tensor product of chain complexes — to be defined!

Proof idea for Theorem 1

(1) Eilenberg–Zilber Thm: $C_\bullet(X \times Y) \simeq C_\bullet(X) \otimes C_\bullet(Y)$

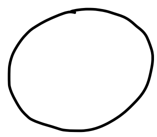
(2) Compute $H_i(C_\bullet(X) \otimes C_\bullet(Y))$ — similar to $H_i(C_\bullet(X) \otimes A)$ for a $\mathbb{Z}$-module A

(12) Twisted Homology (not in exam)

Motivation $K \subseteq S^3$ a knot, ie $K \cong S^1$. Consider the composition

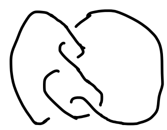
$$\begin{tikzcd} \pi_1(S^3 \setminus K) \arrow[r, "\text{surjection}", "\text{Hurewicz's Theorem}"'] \arrow[rrr, bend right, "p"'] & H_1(S^3 \setminus K) \arrow[r, "\text{iso}", "\text{Alex Duality}"'] & \mathbb{Z} \arrow[r, "\text{surj.}"] & \mathbb{Z}/2 \end{tikzcd}$$

By the classification of coverings, ker $p \subseteq \pi_1(S^3 \setminus K)$ corresponds to a two-sheeted connected covering $M_K \longrightarrow S^3 \setminus K$. What is $H_1(M_K)$? It depends on K!



$H_1(M_K) \cong \mathbb{Z}$

$H_1(M_K) \cong \mathbb{Z} \oplus \mathbb{Z}/3$



$H_1(M_K) \cong \mathbb{Z} \oplus \mathbb{Z}/5$

So these are three distinct knots!

Moreover, it illustrates that homology of coverings of X can be a rich invariant.

Def For a (non-abelian) group G, let the group ring $\mathbb{Z}[G]$ be the free $\mathbb{Z}$-module with basis G, and multiplication

$$\underbrace{\left(\sum_{g \in G} a_g\, g\right)}_{} \cdot \underbrace{\left(\sum_{g \in G} b_g\, g\right)}_{\text{finite } \mathbb{Z}\text{-linear combinations of elements of } G} := \underbrace{\sum_{g,h \in G} a_g b_h\, (gh)}_{}$$

$\mathbb{Z}[G]$ is a unital ring, and commutative if and only if G is abelian.

Ex $\mathbb{Z}[\mathbb{Z}/n] \cong \mathbb{Z}[t]/(t^n - 1)$, $\quad \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$

$\mathbb{Z}[S_3] \cong \mathbb{Z}\langle x, y\rangle / (x^2 - 1,\ y^3 - 1,\ xyxy - 1)$

$Y \xrightarrow{p} X$ a regular covering with deck transformation group G (group of homeos $g : Y \to Y$ with $p = p \circ g$)

G acts from the left on Y.

G also acts from the left on $C_i(Y)$ by $g \cdot \sigma := g \circ \sigma$.

That makes $C_i(Y)$ into a left $\mathbb{Z}[G]$-module.

The differentials of $C_\bullet(Y)$ are $\mathbb{Z}[G]$-linear!

Def We write $C_\bullet^{tw}(X; \mathbb{Z}[G])$ for $C_\bullet(Y; \mathbb{Z})$ with the above left $\mathbb{Z}[G]$-module structure and call this a twisted chain complex.

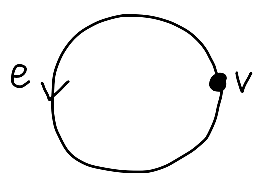
Its n-th homology $H_n^{tw}(X; \mathbb{Z}[G])$ inherits the left $\mathbb{Z}[G]$-module structure!

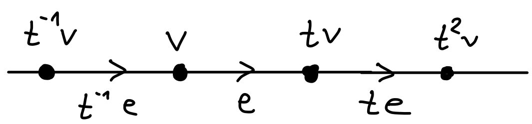
In particular, if X admits a universal covering $\tilde{X}$, we may consider

$$C_\bullet^{tw}(X; \mathbb{Z}[\pi_1 X]).$$

Remark $C_i^{tw}(X; \mathbb{Z}[G])$ is a free $\mathbb{Z}[G]$-module! But $H_i^{tw}(X; \mathbb{Z}[G])$ need not be free.

Ex $C_\bullet^{tw}(S^1; \underbrace{\mathbb{Z}[\pi_1 S^1]}_{\cong \mathbb{Z}[t, t^{-1}]})$ using cellular homology:



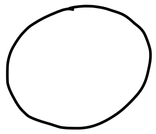


$C_\bullet^{CW}(S^1)$: $\qquad \xleftarrow{\otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1}]/(t-1)} \qquad C_\bullet^{tw, CW}(S^1; \mathbb{Z}[t^{\pm 1}])$

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \qquad\qquad \mathbb{Z}[t^{\pm 1}] \xrightarrow[d_1]{t-1} \mathbb{Z}[t^{\pm 1}]$$

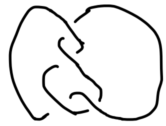
$$\Rightarrow H_1^{tw}(S^1; \mathbb{Z}[t^{\pm 1}]) \cong \ker d_1 = 0$$

$$H_0^{tw}(S^1; \mathbb{Z}[t^{\pm 1}]) \cong \operatorname{coker} d_1 \cong \mathbb{Z}[t^{\pm 1}]/(t-1)$$



$H_1(S^3\setminus K; \mathbb{Z}[t^{\pm1}])$
$\cong 0$

$H_1(S^3\setminus K; \mathbb{Z}[t^{\pm1}])$
$\cong \mathbb{Z}[t^{\pm1}]/(t^{-1}-1+t)$



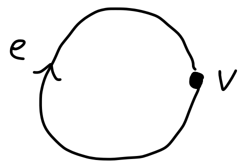
$H_1(S^3\setminus K; \mathbb{Z}[t^{\pm1}])$
$\cong \mathbb{Z}[t^{\pm1}]/(t^{-1}-3+t)$

Theorem (Twisted Poincaré Duality)

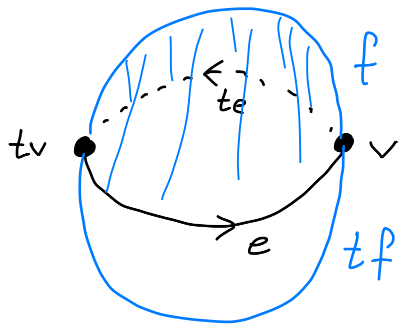
If X is compact and Y orientable, then

$$H^i_{tw}(M; \mathbb{Z}[G]) \cong H^{tw}_i(M; \mathbb{Z}[G])$$

Ex $C^{tw}_\bullet(\mathbb{RP}^2; \mathbb{Z}[\mathbb{Z}/2])$ again using cellular homology.



and a 2-cell f



$$C^{CW}_\bullet(\mathbb{RP}^2) \xleftarrow{\otimes_R R/(t-1)} C^{tw,CW}_\bullet(\mathbb{RP}^2; \underbrace{\mathbb{Z}[t]/(t^2-1)}_{R})$$

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \qquad\qquad R \xrightarrow{t+1} R \xrightarrow{t-1} R$$

$$H^{tw}_0 \cong R/(t-1)$$
$$H^{tw}_1 \cong 0$$
$$H^{tw}_2 \cong (t-1)$$

$C^\bullet_{tw,CW}(\mathbb{RP}^2; R)$ is the dual of $C^{tw,CW}_\bullet$: $R \xleftarrow{t+1} R \xleftarrow{t-1} R$

So $H^0_{tw} \cong (t+1)$, $H^1_{tw} \cong 0$, $H^2_{tw} \cong R/(t+1)$

Indeed, we find $H^{tw}_0 \cong H^2_{tw}$, $H^{tw}_2 \cong H^0_{tw}$, using the iso $R \to R$, $t \mapsto -t$.