Exercise sheet 14

p-adic Numbers, Absolute Values

1. Determine the p-adic expansions of ± 1 and $\frac{\pm 1}{1-p}$ for an arbitrary prime p.

2. Represent the rational numbers $\frac{2}{3}$ and $-\frac{2}{3}$ as 5-adic numbers.

3. (a) Show that a rational number x with $\mathrm{ord}_p(x) = 0$ has a purely periodic p-adic expansion if and only if $x \in [-1, 0)$.

   (b) Show that in $\mathbb{Q}_p$ the numbers with eventually periodic p-adic expansions are precisely the rational numbers.

4. Show that the equation $x^2 = 2$ has a solution in $\mathbb{Z}_7$ and compute its first few 7-adic digits.

5. For which primes p is -1, resp. 2, resp. 3 a square in $\mathbb{Q}_p$?

*6. For any integer $b \geqslant 2$ consider the map

$$\pi\colon \prod_{i\geqslant 1}\{0,1,\ldots,b-1\} \longrightarrow [0,1], \quad (a_i)_i \mapsto \sum_{i\geqslant 1} a_i b^{-i}.$$

Show that π is surjective and determine its fibers. Prove that the natural topology on the interval $[0,1]$ is the quotient topology via π from the product topology on $\prod_{i\geqslant 1}\{0,1,\ldots,b-1\}$, where each factor is endowed with the discrete topology. Interpret this fact by comparing the topologies on the source and the target.

7. Prove that for any prime p the ring of endomorphisms of the additive group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is canonically isomorphic to $\mathbb{Z}_p$.