D-MATH
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Number Theory II

FS 2024

Exercise sheet 15

Valuations, Absolute Values

*1. Let A be a Dedekind ring and π a prime element. Construct an isomorphism between the completion $A_{(\pi)}$ and the ring $A[[X]]/(X-\pi)$.

2. Let K_p denote the field of germs of meromorphic functions near a point $p \in \mathbb{C}$. For any $f \in K_p$ let $\mathrm{ord}_p(f)$ denote the vanishing order, respectively minus the pole order, of f at p, respectively ∞ if $f = 0$. Show that this is a valuation and determine the valuation ring and its maximal ideal. Decide whether this valuation is complete and determine the completion.

   (The elements of K_p can be identified with the Laurent series in $z - p$ with finite principal parts which have a positive radius of convergence.)

3. Let $|\cdot|$ be an absolute value on a field K. Show that $|\cdot|^\alpha$ is also an absolute value for every $0 < \alpha \leqslant 1$.

4. Show that for any absolute value $|\ |$ on a field K, the maps $+, \cdot \colon K \times K \to K$ and $(\ )^{-1} \colon K \smallsetminus \{0\} \to K \smallsetminus \{0\}$ are continuous for the induced topology.

5. Find all primes numbers p such that the sequence $\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots$ converges in $\mathbb{Q}_p$.

6. According to exercise 5 of sheet 3 the ring $A := \mathbb{Z}[\sqrt{-5}\,]$ is a Dedekind ring with the maximal ideal $\mathfrak{p} := (3, 1+\sqrt{-5}\,)$ and $A/\mathfrak{p} \cong \mathbb{F}_3$. Since $2 \in A \smallsetminus \mathfrak{p}$ it follows that 2 becomes a unit in $A_\mathfrak{p}$. Determine its reciprocal $\frac{1}{2} \in A_\mathfrak{p}$ explicitly.