Solutions 15

Valuations, Absolute Values, Power series

*1. Let A be a Dedekind ring and π a prime element. Construct an isomorphism between the completion $A_{(\pi)}$ and the ring $A[[X]]/(X-\pi)$.

Solution: See Proposition 2.6 in Section 2 of Chapter 2 of Neukirch.

2. Let K_p denote the field of germs of meromorphic functions near a point $p \in \mathbb{C}$. For any $f \in K_p$ let $\mathrm{ord}_p(f)$ denote the vanishing order, respectively minus the pole order, of f at p, respectively ∞ if $f = 0$. Show that this is a valuation and determine the valuation ring and its maximal ideal. Decide whether this valuation is complete and determine the completion.

(The elements of K_p can be identified with the Laurent series in $z-p$ with finite principal parts which have a positive radius of convergence.)

Solution: The function ord_p is well-defined on formal Laurent series with finite principal parts and satisfies all the conditions for a valuation on $\mathbb{C}((z-p))$. It therefore also induces a valuation on the subfield K_p. The associated valuation ring $\mathcal{O}_p$ consists of all germs that are holomorphic at p, and the maximal ideal $\mathfrak{m}_p$ consists of all germs that are holomorphic and vanish at p.

Also, for any integer $n \geqslant 0$ the ideal $\mathfrak{m}_p^n$ consists of all power series whose first n coefficients vanish. Thus $\mathcal{O}_p/\mathfrak{m}_p^n \cong \mathbb{C}[z-p]/(z-p)^n$. Passing to the limit the completion is therefore

$$\widehat{\mathcal{O}}_p \;\cong\; \varprojlim_n \mathbb{C}[z-p]/(z-p)^n \;\cong\; \mathbb{C}[[z-p]].$$

As there are formal power series with radius of convergence zero, it follows that $\mathcal{O}_p \to \widehat{\mathcal{O}}_p$ is not an isomorphism. Thus the valuation ord_p is not complete on K_p.

3. Let $|\cdot|$ be an absolute value on a field K. Show that $|\cdot|^\alpha$ is also an absolute value for every $0 < \alpha \leqslant 1$.

Solution: Let $x, y \in K$. Since $|\cdot|$ is an absolute value, we have $|x|^\alpha \geqslant 0$ with equality if and only if $x = 0$. Furthermore $|xy|^\alpha = (|x||y|)^\alpha = |x|^\alpha |y|^\alpha$. Also there exists $z \in K$ with $|z| \notin \{0,1\}$ and hence $|z|^\alpha \notin \{0,1\}$. It remains to show the triangle inequality. For this note that $|\ |^\alpha = h \circ |\ |$ for the function $h\colon [0,\infty) \to [0,\infty)$, $a \mapsto a^\alpha$. Since the second derivative $h''(t) = \alpha(\alpha-1)t^{\alpha-2}$ is

$\leqslant 0$ on the interval $(0,\infty)$, this function is *concave*, i.e., for all $a,b\in[0,\infty)$ and $t\in[0,1]$ we have

$$h(ta+(1-t)b)\geqslant th(a)+(1-t)h(b).$$

Since also $h(0)=0$, using the following lemma from analysis we can conclude that $|x+y|^\alpha\leqslant(|x|+|y|)^\alpha\leqslant|x|^\alpha+|y|^\alpha$, as desired.

Lemma. *Any concave function $f\colon[0,\infty)\to\mathbb{R}$ with $f(0)\geqslant 0$ is* subadditive*, that is, it satisfies $f(a+b)\leqslant f(a)+f(b)$ for all $a,b\in[0,\infty)$.*

Proof. For all $x\in[0,\infty)$ and $t\in[0,1]$ we have

$$f(tx)=f(tx+(1-t)0)\geqslant tf(x)+(1-t)f(0)\geqslant tf(x).$$

For all $a,b\in[0,\infty)$ it follows that

$$\begin{aligned}f(a)+f(b)&=f\left(\frac{a}{a+b}(a+b)\right)+f\left(\frac{b}{a+b}(a+b)\right)\\&\geqslant\frac{a}{a+b}f(a+b)+\frac{b}{a+b}f(a+b)=f(a+b).\end{aligned}$$

□

4. Show that for any absolute value $|\ |$ on a field K, the maps $+,\cdot\colon K\times K\to K$ and $(\ )^{-1}\colon K\smallsetminus\{0\}\to K\smallsetminus\{0\}$ are continuous for the induced topology.

   Solution: By definition K is a metric space and therefore first countable; hence so is $K\times K$ with the product topology. Thus it suffices to verify the sequential criterion for continuity in all cases. Also, a sequence $(x_n,y_n)_{n\geqslant 0}$ in $K\times K$ converges to (x,y) if and only if the sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ in K converge to x and y, respectively.

   So consider such sequences. Then by the triangle inequality we have

$$|(x_n+y_n)-(x+y)|\leqslant|x_n-x|+|y_n-y|\xrightarrow{n\to\infty}0,$$

   hence the sequence $(x_n+y_n)_{n\geqslant 0}$ converges to $x+y$. Thus addition is continuous. Furthermore

$$\begin{aligned}|x_ny_n-xy|&=|(x_n-x)(y_n-y)+(x_n-x)y+x(y_n-y)|\\&\leqslant|(x_n-x)(y_n-y)|+|(x_n-x)y|+|x(y_n-y)|\\&=|x_n-x||y_n-y|+|y||x_n-x|+|x||y_n-y|\xrightarrow{n\to\infty}0,\end{aligned}$$

   hence the sequence $(x_ny_n)_{n\geqslant 0}$ converges to xy. Thus multiplication is continuous. Finally suppose that $x\neq 0$. Then $|x_n|\to|x|>0$ because $|\ |\colon K\to\mathbb{R}$ is continuous, so $x_n\neq 0$ for all $n\gg 0$ and $|x_n|^{-1}$ remains bounded for $n\to\infty$. Thus

$$|x_n^{-1}-x^{-1}|=|x_n^{-1}x^{-1}||x-x_n|=|x_n|^{-1}|x|^{-1}|x-x_n|\xrightarrow{n\to\infty}0,$$

   hence the sequence $(x_n^{-1})_{n\geqslant 0}$ converges to x^{-1}. Thus the inverse is continuous.

5. Find all primes numbers p such that the sequence $\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots$ converges in $\mathbb{Q}_p$.

Solution: The difference of any two adjacent terms is $10^{-n} - 10^{-n-1} = 9 \cdot 10^{-n-1}$, and regardless of the choice of p we always have $\mathrm{ord}_p(9 \cdot 10^{-n-1}) \leqslant 2$. Thus these differences do not converge to 0 in $\mathbb{Q}_p$ for $n \to \infty$. Therefore the original sequence is not a Cauchy sequence in $\mathbb{Q}_p$ and thus does not converge. So the answer is the empty set.

6. According to exercise 5 of sheet 3 the ring $A := \mathbb{Z}[\sqrt{-5}\,]$ is a Dedekind ring with the maximal ideal $\mathfrak{p} := (3, 1+\sqrt{-5}\,)$ and $A/\mathfrak{p} \cong \mathbb{F}_3$. Since $2 \in A \smallsetminus \mathfrak{p}$ it follows that 2 becomes a unit in $A_\mathfrak{p}$. Determine its reciprocal $\frac{1}{2} \in A_\mathfrak{p}$ explicitly.

Solution: We formally compute

$$\frac{1}{2} \;=\; \frac{2}{4} \;=\; \frac{2}{1-(-3)} \;=\; \sum_{n\geqslant 0} 2\cdot(-3)^n,$$

which makes sense and is correct within $A_\mathfrak{p}$, because -3 lies in $\mathfrak{p}$ and therefore also in the maximal ideal of $A_\mathfrak{p}$. Moreover we have $(-3)^n \in \mathfrak{p}^k$ for all $n \geqslant k \geqslant 0$; hence the finite geometric sum

$$a_k \;:=\; \sum_{n=0}^{k-1} 2\cdot(-3)^n \;=\; 2\cdot\frac{1-(-3)^k}{1-(-3)} \;=\; \frac{1-(-3)^k}{2}$$

represents the reciprocal of 2 in $A/\mathfrak{p}^k$. Thus $\frac{1}{2} = (a_k + \mathfrak{p}^k)_k \in \varprojlim_n A/\mathfrak{p}^n = A_\mathfrak{p}$.

Fun Fact: This has nothing to do with the special situation and holds for any maximal ideal $\mathfrak{p}$ of any ring A with residue field of characteristic 3. Doing mathematics successfully sometimes involves distinguishing relevant information from irrelevant one.