Exercise sheet 16

Absolute Values, Completion, Power series

1. *(Product formula)* A non-archimedean absolute value $|\ |$ on a field K, whose valuation ring $\mathcal{O}_K$ is discrete with finite residue field $\mathcal{O}_K/\mathfrak{m}$, is called *normalized* if $|\pi| = |\mathcal{O}_K/\mathfrak{m}|^{-1}$ for any element π with $(\pi) = \mathfrak{m}$. Consider a finite field k.

   (a) Write down all normalized absolute values $|\ |_v$ on $k(t)$.

   (b) For any $a \in k(t)^\times$ prove that $\prod_v |a|_v = 1$.

   (*Hint:* Compare Examples 8.2.6 (a–b) and Theorem 8.4.15 of Ostrowski.)

2. Work out the details of the proof of Proposition 8.5.5 of the lecture: Every metric space possesses a completion.

3. Let K be a complete ultrametric field. Show that a convergent series with summands in K can be arbitrarily rearranged and subdivided without changing convergence or the limit.

   (*Hint:* Test your analysis skills by trying to give a complete formal proof.)

4. Let K be a field with a complete absolute value $|\ |$. The *radius of convergence* of a power series $f(X) = \sum_{n=0}^\infty a_n X^n \in K[[X]]$ is defined as

$$r_f \ := \ \sup\bigl\{r \in \mathbb{R}^{\geqslant 0} : |a_n| r^n \to 0 \text{ for } n \to \infty\bigr\} \ \in \ \mathbb{R} \cup \{\infty\}.$$

   (a) Show that

$$r_f \ = \ \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

   (b) Show that for any $x \in K$ the series $f(x) := \sum_{n=0}^\infty a_n x^n$ diverges if $|x| > r_f$ and converges if $|x| < r_f$.

   (c) What happens for $|x| = r_f$?

Please turn over

5. Let K be a field that is complete with respect to a p-adic absolute value. Consider $\alpha, \beta \in \mathbb{Z}_p$ and $m, n \in \mathbb{Z}$ with $n \geqslant 0$. Prove:

(a) The binomial coefficient $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ lies in $\mathbb{Z}_p$.

(b) The power series $F_\alpha(X) := \sum_{n\geqslant 0} \binom{\alpha}{n} X^n \in K[[X]]$ has convergence radius $\geqslant 1$. Moreover, for $x \in K$ with $|x| < 1$ we have $|F_\alpha(x) - 1| < 1$.

(c) $F_{\alpha+\beta}(x) = F_\alpha(x) \cdot F_\beta(x)$.

(d) $F_{m\alpha}(x) = F_\alpha(x)^m$.

(e) $F_m(x) = (1 + x)^m$.

(f) $y := F_{m/n}(x)$ is the only solution of the equation $y^n = (1+x)^m$ with $|y-1| < 1$, if $p \nmid n$.

This therefore justifies writing $F_\alpha(x) = (1 + x)^\alpha$.

*(g) Do we then also have $((1 + x)^\alpha)^\beta = (1 + x)^{\alpha\beta}$?

(h) Find a closed form of $\sqrt{7}$ in $\mathbb{Q}_3$.

*6. *(Newton method for finding zeros of a polynomial)* Let p be a prime number, let $f \in \mathbb{Z}_p[X]$ and let $\alpha \in \mathbb{Z}_p$ be a root of f such that $f'(\alpha) \neq 0$. Set

$$U := \{a \in \mathbb{Z}_p : |f(a)| < |f'(a)|^2 \text{ and } |\alpha - a| < |f'(a)|\},$$

which is an open neighborhood of α in $\mathbb{Z}_p$. Take $a_1 \in U$ and recursively define $a_{n+1} := a_n - \frac{f(a_n)}{f'(a_n)}$ for $n \geqslant 1$. Show that for all n:

(a) $a_n \in U$,

(b) $|f'(a_n)| = |f'(a_1)|$,

(c) $|f(a_n)| \leqslant |f'(a_1)|^2 t^{2^{n-1}}$ for $t = |f(a_1)/f'(a_1)| < 1$.

Moreover, show that $\lim_{n\to\infty} a_n = \alpha$ and $|f'(\alpha)| = |f'(a_1)|$.