Exercise sheet 17

Absolute values, Extensions of Complete Absolute Values

1. Let $|\ |$ be the usual archimedean absolute value on $\mathbb{R}$ and on $\mathbb{Q}$.

   (a) Prove that $\|(x,y)\| := |x+\sqrt{2}\,y|$ defines a norm on the $\mathbb{Q}$-vector space $\mathbb{Q}^2$, which is not equivalent to the euclidean norm.

   (b) Can one construct a similar example with the p-adic norm on $\mathbb{Q}$?

2. Determine to which extent the factors in Hensel's lemma are unique.

*3. Here we consider $\mathbb{Q}_p$ as an abstract field and include $\mathbb{Q}_\infty := \mathbb{R}$.

   (a) Show that $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic for any $p \neq q$.

   (b) Prove that every automorphism of $\mathbb{Q}_p$ is trivial.

   Hint: Look at which elements are squares in the respective field.

4. Prove that every finite extension of $\mathbb{C}((t))$ of degree n is isomorphic to $\mathbb{C}((s))$ where $s^n = t$.

5. Let K be a non-archimedean complete field such that $\mathcal{O}_K$ is a discrete valuation ring. Prove that for every finite extension L/K with separable residue field extension there exists $\alpha \in L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

6. Let K be a field with a complete discrete valuation v, and let $\bar{K}$ be an algebraic closure of K. In the lecture we have seen that v extends uniquely to a valuation $\bar{v}$ on $\bar{K}$. Show that this extension is not complete.

   Hint: Consider roots of an element in K with positive valuation.