Exercise sheet 18

Newton Polygons, Extensions of Absolute Values

1. (a) Show that $X^3 - X^2 - 2X - 8$ is irreducible in $\mathbb{Q}[X]$ but splits completely in $\mathbb{Q}_2[X]$.

   (b) Find two monic polynomials of degree 3 in $\mathbb{Q}_5[X]$ with the same Newton polygons, but one irreducible and the other not.

   (c) Hensel's lemma concerns a polynomial f with a factorization $(f \bmod \mathfrak{p}) = \bar{g}\bar{h}$ such that $\bar{g}$ and $\bar{h}$ are coprime. Show by a counterexample that the assumption 'coprime' is necessary.

2. *(Krasner's lemma)* Let K be a field that is complete for a non-archimedean absolute value $|\ |$. Let $|\ |$ also denote the unique extension to an algebraic closure $\bar{K}$. Consider an element $\alpha \in \bar{K}$ that is separable over K, and let $\alpha = \alpha_1, \ldots, \alpha_n$ be its Galois conjugates over K. Consider an element $\beta \in \bar{K}$ such that

   $$|\alpha - \beta| < |\alpha - \alpha_i|$$

   for all $2 \leqslant i \leqslant n$. Show that $K(\alpha) \subseteq K(\beta)$.

   Hint: Let M be the Galois closure of the extension $K(\alpha, \beta)/K(\beta)$ and consider the action of $\mathrm{Gal}(M/K(\beta))$ on α.

*3. Consider an integer $n \geqslant 1$ and a finite set S of rational primes $p \leqslant \infty$ (including $\mathbb{Q}_\infty = \mathbb{R}$). For each $p \in S$ consider field extensions $L_{p,i}/\mathbb{Q}_p$ for $1 \leqslant i \leqslant r_p$ such that $\sum_{i=1}^{r_p} [L_{p,i}/\mathbb{Q}_p] = n$. Show that there exists a number field L of degree n over $\mathbb{Q}$ such that for every $p \in S$ we have $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^{r_p} L_{p,i}$.

   Hint: Use Krasner's lemma from above or adapt it suitably.

4. Let L/K be a purely inseparable finite extension of degree q. Show that every absolute value $|\ |$ on K possesses a unique extension to L, given by the formula

   $$|y| \ := \ |y^q|^{1/q}.$$

*5. Let L/K be a finite field extension and let $|\ |$ be a (nontrivial) absolute value on L. Show that the restriction of $|\ |$ to K is nontrivial.

   (*Hint:* Use Newton polygons.)

6. (a) Determine all the absolute values on $\mathbb{Q}(\sqrt{5}\,)$.

   (b) How many extensions to $\mathbb{Q}(\sqrt[n]{2}\,)$ does the archimedean absolute value on $\mathbb{Q}$ admit?