Solutions 19

Extensions of Absolute Values, Local and Global Fields

1. Consider a Dedekind ring A with a maximal ideal $\mathfrak{p}$. Let L be a finite Galois extension of $K := \mathrm{Quot}(A)$ with Galois group Γ. Let B be the integral closure of A in L and let $\mathfrak{q}$ be a prime of B above $\mathfrak{p}$. Let K' be the intermediate field corresponding to the decomposition group $\Gamma_\mathfrak{q}$, and consider the prime ideal $\mathfrak{p}' := \mathfrak{q} \cap K'$ of $A' := B \cap K'$. Prove that the inclusion $A \hookrightarrow A'$ induces an isomorphism of completions $A_\mathfrak{p} \xrightarrow{\sim} A'_{\mathfrak{p}'}$.

Solution: By Propositions 9.5.2 and 9.5.6 for L/K we have natural isomorphisms

$$(1) \qquad L \otimes_K \hat{K} \;\cong\; \bigtimes_{i=1}^{r} \hat{L}_i \qquad \text{and} \qquad B \otimes_A \mathcal{O} \;\cong\; \bigtimes_{i=1}^{r} \mathcal{O}_i$$

for finite separable field extensions $\hat{L}_i/\hat{K}$ with $[L/K] = \sum_{i=1}^r [\hat{L}_i/\hat{K}]$, where $\mathcal{O} = A_\mathfrak{p}$. Moreover, by Propositions 9.5.4 and 9.5.7 (a) and 9.5.10, the prime ideals of B above $\mathfrak{p}$ are precisely the r different ideals $\mathfrak{q}_i := \mathfrak{n}_i \cap B$ with the associated completion $\hat{L}_i$, which is galois over $\hat{K}$ with Galois group $\Gamma_{\mathfrak{q}_i}$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{q}_1$.

Then the factors $\hat{L}_1$ and $\mathcal{O}_1$ in the cartesian products in (1) are stable under $\Gamma_\mathfrak{q}$ with $\hat{L}_1^{\Gamma_\mathfrak{q}} = \hat{K}$ and hence $\mathcal{O}_1^{\Gamma_\mathfrak{q}} = \mathcal{O}_1 \cap \hat{K} = \mathcal{O}$. On the other hand we have $L^{\Gamma_\mathfrak{q}} = K'$ and hence $B^{\Gamma_\mathfrak{q}} = B \cap K' = A'$. Taking $\Gamma_\mathfrak{q}$-invariants† in (1) thus shows that

$$(2) \qquad K' \otimes_K \hat{K} \;\cong\; \hat{K} \times (\text{other factors}) \quad \text{and} \quad A' \otimes_A \mathcal{O} \;\cong\; \mathcal{O} \times (\text{other factors}).$$

By Propositions 9.5.2 and 9.5.6 for the extension K'/K the factors $\hat{K}$ and $\mathcal{O}$ on the right hand sides of (2) must therefore be the completions of K' and A' with respect to a certain prime ideal of A' above $\mathfrak{p}$. The inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_i$ shows that this can only be the prime ideal $\mathfrak{p}' := \mathfrak{q} \cap K'$.

†: For any K-vector space V with an action of a group G and any overfield $\hat{K}/K$ there is a natural isomorphism $(V \otimes_K \hat{K})^G \cong V^G \otimes_K \hat{K}$. To see this choose a basis $\mathcal{B}$ of $\hat{K}$ over K, which induces an isomorphism of K-vector spaces $K^{(\mathcal{B})} \cong \hat{K}$. This then induces natural isomorphisms

$$(V \otimes_K \hat{K})^G \cong (V \otimes_K K^{(\mathcal{B})})^G \cong (V^{(\mathcal{B})})^G \cong (V^G)^{(\mathcal{B})} \cong V^G \otimes_K K^{(\mathcal{B})} \cong V^G \otimes_K \hat{K}.$$

In particular this yields the first isomorphism in (2). The second follows from the first by intersecting with $B \otimes_A \mathcal{O}$.

2. Show that any local field is the completion of a global field at an absolute value.

   Solution: By definition the local fields are, up to isomorphism, the finite extensions of $\mathbb{R}$, $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$.

   The archimedean complete fields $\mathbb{R}$ and $\mathbb{C}$ are the completions of $\mathbb{Q}$ and $\mathbb{Q}(i)$ with respect to the usual archimedean absolute value.

   By Proposition 11.1.4 (b) of the lecture any local field of positive characteristic is isomorphic to $k((t))$ for a finite field k. This is the completion of the global field $k(t)$ for the valuation ord_t.

   Suppose now that $K = \mathbb{Q}_p(\alpha)$ is a finite extension of $\mathbb{Q}_p$. Let f be the minimal polynomial of α over $\mathbb{Q}_p$ with zeros $\alpha, \alpha_2, \ldots, \alpha_n$. As in the solution of exercise 3 of sheet 18, we can choose a monic polynomial $g \in \mathbb{Q}[X]$ of degree n that is coefficientwise close to f and has a root β such that

$$|\alpha - \beta| < |\min\{|\alpha - \alpha_i \mid 2 \leqslant i \leqslant n\}.$$

   As in the solution, we get $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta)$. Thus K is the completion of the number field $\mathbb{Q}(\beta)$ with respect to $|\ |_p$.