D-MATH
Prof. Richard Pink

Number Theory II

FS 2024

Exercise sheet 20

Profinite Groups, Infinite Galois Theory

1. Consider a topological group G.

   (a) Show that if G is hausdorff, then the center of G and the centralizer of any element $g \in G$ are closed subgroups.

   (b) Show that for any continuous action of G on a topological space the stabilizer of any closed point is closed.

   (c) Show that G is hausdorff if and only if G is T_0. (A topological space is called T_0 if for any two distinct points, one of them possesses a neighborhood that does not contain the other.)

*2. A topological space is called *totally disconnected* if every connected subset contains only one element. Prove that a topological group is profinite if and only if it is compact and totally disconnected.

3. Consider a Galois extension L/K with $\Gamma := \mathrm{Gal}(L/K)$ and an intermediate field K' with $\Gamma' := \mathrm{Gal}(L/K')$. Show that K'/K is Galois if and only if $\Gamma' \triangleleft \Gamma$, and that then there is a natural isomorphism of profinite groups $\Gamma/\Gamma' \cong \mathrm{Gal}(K'/K)$.

4. *(The cyclotomic $\mathbb{Z}_p$-extension)* Set $\mathbb{Q}(\mu_{p^\infty}) := \bigcup_n \mathbb{Q}(\mu_{p^n})$ for a prime number p.

   (a) Show that $\mathbb{Q}(\mu_{p^\infty})$ possesses a unique subfield K_∞ with $\mathrm{Gal}(K_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$.

   *(b) Give explicit generators for K_∞.

5. Let p be a prime number and $\bar{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$.

   (a) Show that $|\ |_p$ extends to some absolute value $|\ |$ on $\bar{\mathbb{Q}}$.

   (b) For any subfield $K \subset \bar{\mathbb{Q}}$ which is finite over $\mathbb{Q}$ let $\hat{K}$ be the completion of K with respect to the restriction of $|\ |$. Show that for any subfields $K \subset L \subset \bar{\mathbb{Q}}$ which are finite over $\mathbb{Q}$ we get a natural inclusion $\hat{K} \hookrightarrow \hat{L}$.

   (c) Show that the union $\bar{\mathbb{Q}}_p$ of all these $\hat{K}$ is an algebraic closure of $\mathbb{Q}_p$.

   (d) Show that there is a natural isomorphism

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Stab}_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(|\ |).$$