Solutions 21

Structure of Local Fields, Unramified and Tame Extensions

1. Let K be a nonarchimedean local field of characteristic $p > 0$ with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Show that the subgroup $U_1 := \{x \in \mathcal{O}^\times \mid x \equiv 1 \bmod \mathfrak{m}\}$ is topologically isomorphic to a countably infinite product of copies of $\mathbb{Z}_p$.

Solution Choose a uniformizer $\pi \in \mathcal{O}$ and a basis $\alpha_1, \ldots, \alpha_n$ of the residue field $\mathcal{O}/\mathfrak{m}$ over $\mathbb{F}_p$. We claim that we have a topological isomorphism

$$\bigtimes_{i=1}^{n} \bigtimes_{p\nmid j>0} \mathbb{Z}_p \xrightarrow{\sim} U_1, \quad (m_{i,j})_{i,j} \mapsto \prod_{i=1}^{n} \prod_{p\nmid j>0} (1+\alpha_i\pi^j)^{m_{i,j}}.$$

For this note first that for any $k \geqslant 0$, all but finitely many factors on the right hand side are $\equiv 1 \bmod \mathfrak{m}^k$. Thus the product is well-defined. Moreover, by the binomial series (compare exercise 5 of sheet 16) the map is a continuous homomorphism in each factor $\mathbb{Z}_p$. Thus altogether it is a continuous homomorphism. Once we have proved that it is bijective, being a continuous bijection between compact Hausdorff spaces it is then a homeomorphism and therefore a topological group isomorphism.

To prove the bijectivity, for any integer $m > 0$ consider the subgroup

$$U_m \;:=\; \{x \in \mathcal{O}^\times \mid x \equiv 1 \bmod \mathfrak{m}^m\}.$$

We set $I := \{0, 1, \ldots, p-1\}$ and note that $I^{\mathbb{N}} \to \mathbb{Z}_p$, $(b_k)_\nu \mapsto \sum_{k=0}^{\infty} b_k p^k$ is bijective. Thus it suffices to show that the map

$$\bigtimes_{i=1}^{n} \bigtimes_{p\nmid j>0} \bigtimes_{k=0}^{\infty} I \longrightarrow U_1, \quad (b_{i,j,k})_{i,j,k} \mapsto \prod_{i=1}^{n} \prod_{p\nmid j>0} \prod_{k=0}^{\infty} (1+\alpha_i\pi^j)^{b_{i,j,k}p^k}$$

is bijective. In this product $m := jp^k$ runs through all positive integers, and for fixed $m = jp^k$ we have

$$\prod_{i=1}^{n} (1+\alpha_i\pi^j)^{b_{i,j,k}p^k} \;=\; \prod_{i=1}^{n} (1+\alpha_i^{p^k}\pi^{jp^k})^{b_{i,j,k}} \;\equiv\; 1 + \left(\sum_{i=1}^{n} \alpha_i^{p^k} b_{i,j,k}\right) \cdot \pi^m \bmod \mathfrak{m}^{m+1}.$$

Here the $\alpha_i^{p^k}$ for $1 \leqslant i \leqslant n$ again form a basis of $\mathcal{O}/\mathfrak{m}$ over $\mathbb{F}_p$; hence the value of the big parenthesis on the right hand side runs through all elements of $\mathcal{O}/\mathfrak{m}$ as $(b_{i,j,k})_i$ varies over $\bigtimes_{i=1}^{n} I$. The total value thus runs through a system of representatives of U_m/U_{m+1}. By induction on m we can thus deduce that

$$\bigtimes_{i=1}^{n} \bigtimes_{\substack{p\nmid j>0 \\ k\geqslant 0 \\ jp^k \leqslant m}} I \longrightarrow U_1/U_{m+1}, \quad (b_{i,j,k})_{i,j,k} \mapsto \prod_{i=1}^{n} \prod_{\substack{p\nmid j>0 \\ k\geqslant 0 \\ jp^k \leqslant m}} (1+\alpha_i\pi^j)^{b_{i,j,k}p^k}$$

is bijective for every $m \geqslant 0$. Taking the limit over m we find the desired bijectivity.

2. Let $\mathbb{Q}_p^{\rm nr}$ be the union of all finite unramified extensions of $\mathbb{Q}_p$. Since this is an algebraic extension of $\mathbb{Q}_p$, the valuation ord_p on $\mathbb{Q}_p$ extends uniquely to a valuation on $\mathbb{Q}_p^{\rm nr}$. Show that this extension is not complete.

   Solution: Choose a prime $\ell \neq p$ and primitive ℓ^n-th roots of unity $\zeta_n \in \bar{\mathbb{Q}}_p$ such that $\zeta_{n+1}^\ell = \zeta_n$ for all $n \geqslant 0$. Then each ζ_n is a root of the monic polynomial $X^{\ell^n} - 1 \in \mathbb{Z}_p[X]$, which is separable modulo (p). Thus $\zeta_n \in \mathbb{Q}_p^{\rm nr}$.

   Now suppose that $\mathbb{Q}_p^{\rm nr}$ is complete. Then $\mathbb{Q}_p^{\rm nr}$ contains the element

$$\xi \ := \ \sum_{n\geqslant 0} \zeta_n p^n.$$

   In other words $K := \mathbb{Q}_p(\xi)$ is a finite unramified extension of $\mathbb{Q}_p$, say of degree d. Its residue field then has order p^d. Let ℓ^m be the maximal power of ℓ that divides $p^d - 1$. Then K contains all ℓ^m-th roots of unity of $\bar{\mathbb{Q}}_p$, but no primitive ℓ^{m+1}-th root of unity. In particular K contains ζ_n for all $n \leqslant m$ and therefore also the element

$$\xi_m \ := \ \Big(\xi - \sum_{n=0}^{m} \zeta_n p^n\Big)/p^{m+1} \ = \ \sum_{n\geqslant 0} \zeta_{n+m+1} p^n.$$

   Its residue field thus contains the residue class ζ_{m+1} of ξ_m, which is a primitive ℓ^{m+1}-th root of unity. This is a contradiction; hence $\mathbb{Q}_p^{\rm nr}$ is not complete.

3. Let L be an algebraic extension of a nonarchimedean local field K.

   (a) Show that there is a maximal intermediate field M that is unramified over K.

   (b) Show that the residue field extension of M/L is trivial.

   Solution: See [Neukirch, Ch.II Prop.7.5.]

4. Show that every finite extension of $\mathbb{C}((t))$ has the form $\mathbb{C}((\sqrt[n]{t}\,))$ for some $n \geqslant 1$.

   Solution: Direct adaptation of the proof of Proposition 11.3.3 of the lecture.