Exercise sheet 24

Abelian Extensions, Group cohomology

1. For an integer $n \geqslant 2$ let L be the maximal abelian extension of $\mathbb{Q}$ for which $\mathrm{Gal}(L/\mathbb{Q})$ has exponent dividing n. Determine $\mathrm{Gal}(L/\mathbb{Q})$ up to isomorphism.

2. Is there an abelian extension $K/\mathbb{Q}$ of degree 2023 that is unramified at all primes not dividing 2024, or vice versa?

3. Show that, up to isomorphism, the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ from exercise 4 of sheet 20 is the unique Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}_p$.

4. Consider a finite cyclic group G of order n and a $\mathbb{Z}[G]$-module M. Take $i \in \{0, -1\}$.

   (a) Show that $\hat{H}^i(G, M)$ is annihilated by n.

   (b) Show that $\hat{H}^i(G, M)$ is finite if M is finitely generated.

5. Prove the *Normal Basis Theorem* for an arbitrary finite Galois extension L/K: There exists $b \in L$ such that the elements ${}^\gamma b$ for $\gamma \in \mathrm{Gal}(L/K)$ form a basis of L over K.