Exercise sheet 25

LOCAL CLASS FIELD THEORY

1. Determine the group of norms $\mathrm{Nm}_{K/\mathbb{Q}_2} K^\times$ for

   (a) $K = \mathbb{Q}_2(\sqrt{-1})$.

   (b) $K = \mathbb{Q}_2(\sqrt{2})$.

2. Set $K := \mathbb{Q}_p$ and $L := \mathbb{Q}_p(\mu_p)$ for an odd prime p.

   (a) Determine the group of norms $\mathrm{Nm}_{L/K} L^\times < K^\times$.

   (b) Express the reciprocity isomorphism $\mathrm{Gal}(L/K) \cong K^\times / \mathrm{Nm}_{L/K} L^\times$ from local class field theory in terms of the cyclotomic character $\mathrm{Gal}(L/K) \cong \mathbb{F}_p^\times$.

3. Consider a local field K, a finite cyclic extension L/K, and a finite abelian extension M/L. Let σ be a generator of $\mathrm{Gal}(L/K)$.

   (a) Show that M/K is Galois if and only if $^\sigma(\mathrm{Nm}_{M/L} M^\times) = \mathrm{Nm}_{M/L} M^\times$.

   (b) Show that M/K is abelian if and only if $\{^\sigma b/b : b \in L^\times\} \subset \mathrm{Nm}_{M/L} M^\times$.