Exercise sheet 26

Local and Global Class Field Theory

1. Let K be a nonarchimedean local field. From Corollary 12.4.4 (a) we know that the map $L \mapsto \mathcal{N}_L := \mathrm{Nm}_{L/K} L^\times$ is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of $K^\times$. We also showed that $L_1 \subset L_2 \iff \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2}$. Prove the remaining parts of Corollary 12.4.4 (b), that is, the formulas

$$\begin{aligned} \mathcal{N}_{L_1 L_2} &= \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \quad \text{and} \\ \mathcal{N}_{L_1 \cap L_2} &= \mathcal{N}_{L_1} \mathcal{N}_{L_2}. \end{aligned}$$

2. Let M_K be the set of places of a global field K, and let S_∞ be the subset of all archimedean places. The ring of *adeles of* K (this is a contraction of "*additive ideles*") is the subring

$$\mathbb{A}_K := \Big\{ (a_v)_v \in \bigtimes_{v \in M_K} K_v \;\Big|\; \forall' v \colon a_v \in \mathcal{O}_{K,v} \Big\}.$$

It is endowed with the topology for which the subrings

$$\mathbb{A}_K^S := \Big\{ (a_v)_v \in \bigtimes_{v \in M_K} K_v \;\Big|\; \forall v \notin S \colon a_v \in \mathcal{O}_{K,v} \Big\} \cong \bigtimes_{v \in S} K_v \times \bigtimes_{v \in M_K \smallsetminus S} \mathcal{O}_{K,v}$$

for all finite subsets $S \subset M_K$ with $S_\infty \subset S$ are open and carry the product topology. We identify K with its image in $\mathbb{A}_K$ under the diagonal embedding $x \mapsto (x, x, \ldots)$ and any K_v with its image under $x_v \mapsto (1, \ldots, 1, x_v, 1, \ldots)$.

(a) Show that for any finite extension L/K, there is a natural topological isomorphism $\mathbb{A}_L \cong \mathbb{A}_K \otimes_K L$ with respect to the topology on $\mathbb{A}_K \otimes_K L \cong (\mathbb{A}_K)^n$ induced by any ordered basis of L over K.

(b) Show that K is discrete and cocompact in $\mathbb{A}_K$.

(c) Show that for any fixed place $v \in M_K$, the subring $K \cdot K_v$ is dense in $\mathbb{A}_K$. (This property is called *strong approximation.*)

(d) Show that the group of ideles I_K is topologically isomorphic to the group of units $\mathbb{A}_K^\times$ with the topology induced from the embedding

$$\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K, \quad \underline{a} \mapsto (\underline{a}, \underline{a}^{-1}).$$

(e) Does the analogue of (c) hold for I_K, that is, is the subgroup $K^\times \cdot K_v^\times$ dense in I_K for any place $v \in M_K$?

3. Let K be a finite extension of $\mathbb{F}_p(t)$. Let M_K denote the set of normalized valuations on K and let k_v denote the residue field at $v \in M_K$. Prove the *product formula* for all $x \in K^\times$:

$$\prod_{v \in M_K} |k_v|^{-v(x)} = 1.$$