D-MATH Number Theory II FS 2024
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Solutions 26

Local and Global Class Field Theory

1. Let K be a nonarchimedean local field. From Corollary 12.4.4 (a) we know that the map $L \mapsto \mathcal{N}_L := \mathrm{Nm}_{L/K} L^\times$ is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of $K^\times$. We also showed that $L_1 \subset L_2 \iff \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2}$. Prove the remaining parts of Corollary 12.4.4 (b), that is, the formulas

$$\begin{aligned} \mathcal{N}_{L_1L_2} &= \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \quad \text{and} \\ \mathcal{N}_{L_1\cap L_2} &= \mathcal{N}_{L_1}\mathcal{N}_{L_2}. \end{aligned}$$

Solution By the functoriality of the reciprocity isomorphisms we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(L_1L_2/K) & \hookrightarrow & \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K) \\ \| \wr & & \| \wr \qquad\qquad \| \wr \\ K^\times/\mathcal{N}_{L_1L_2} & \longrightarrow & K^\times/\mathcal{N}_{L_1} \times K^\times/\mathcal{N}_{L_2}. \end{array}$$

Thus the lower map is injective, and so $\mathcal{N}_{L_1L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}$.

Next the subgroup $\mathcal{N}_{L_1}\mathcal{N}_{L_2}$ is closed of finite index and therefore equal to $\mathcal{N}_L$ for some abelian extension L/K. The inclusion $\mathcal{N}_{L_i} \subset \mathcal{N}_{L_1}\mathcal{N}_{L_2} = \mathcal{N}_L$ then implies that $L \subset L_i$ for every $i \in \{1,2\}$ and therefore $L \subset L_1 \cap L_2$. Conversely the inclusion $L_1 \cap L_2 \subset L_i$ implies that $\mathcal{N}_{L_1\cap L_2} \supset \mathcal{N}_{L_i}$ for every $i \in \{1,2\}$ and hence $\mathcal{N}_{L_1\cap L_2} \supset \mathcal{N}_{L_1}\mathcal{N}_{L_2} = \mathcal{N}_L$, which in turn implies that $L_1 \cap L_2 \subset L$. Thus $L_1 \cap L_2 = L$ and therefore $\mathcal{N}_{L_1\cap L_2} = \mathcal{N}_L = \mathcal{N}_{L_1}\mathcal{N}_{L_2}$.

2. Let M_K be the set of places of a global field K, and let S_∞ be the subset of all archimedean places. The ring of *adeles of K* (this is a contraction of "*additive ideles*") is the subring

$$\mathbb{A}_K := \Big\{ (a_v)_v \in \bigtimes_{v\in M_K} K_v \;\Big|\; \forall' v\colon a_v \in \mathcal{O}_{K,v} \Big\}.$$

It is endowed with the topology for which the subrings

$$\mathbb{A}_K^S := \Big\{ (a_v)_v \in \bigtimes_{v\in M_K} K_v \;\Big|\; \forall v \notin S\colon a_v \in \mathcal{O}_{K,v} \Big\} \cong \bigtimes_{v\in S} K_v \times \bigtimes_{v\in M_K\smallsetminus S} \mathcal{O}_{K,v}$$

for all finite subsets $S \subset M_K$ with $S_\infty \subset S$ are open and carry the product topology. We identify K with its image in $\mathbb{A}_K$ under the diagonal embedding $x \mapsto (x, x, \ldots)$ and any K_v with its image under $x_v \mapsto (1, \ldots, 1, x_v, 1, \ldots)$.

(a) Show that for any finite extension L/K, there is a natural topological isomorphism $\mathbb{A}_L \cong \mathbb{A}_K \otimes_K L$ with respect to the topology on $\mathbb{A}_K \otimes_K L \cong (\mathbb{A}_K)^n$ induced by any ordered basis of L over K.

(b) Show that K is discrete and cocompact in $\mathbb{A}_K$.

(c) Show that for any fixed place $v \in M_K$, the subring $K \cdot K_v$ is dense in $\mathbb{A}_K$. (This property is called *strong approximation.*)

(d) Show that the group of ideles I_K is topologically isomorphic to the group of units $\mathbb{A}_K^\times$ with the topology induced from the embedding

$$\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K, \quad \underline{a} \mapsto (\underline{a}, \underline{a}^{-1}).$$

(e) Does the analogue of (c) hold for I_K, that is, is the subgroup $K^\times \cdot K_v^\times$ dense in I_K for any place $v \in M_K$?

Solution

(a) Choose an ordered basis $b_1, \ldots, b_n$ of L over K. Then by Proposition 9.5.2 of the lecture, for every place $v \in M_K$ we have isomorphisms

$$\begin{array}{ccccc} K_v^n & \xrightarrow{\sim} & K_v \otimes_K L & \xrightarrow{\sim} & \times_{w|v} L_w \\ (a_{v,i})_i & \longmapsto & \sum_i a_{v,i} \otimes b_i & \longmapsto & \left(\sum_i a_{v,i} b_i\right)_w. \end{array} \tag{1}$$

Taking the product over all v this induces isomorphisms

$$\underset{v \in M_K}{\times} K_v^n \xrightarrow{\sim} \underset{v \in M_K}{\times} K_v \otimes_K L \xrightarrow{\sim} \underset{w \in M_L}{\times} L_w. \tag{2}$$

Now let S_1 be the finite set of all $v \in M_K$ which are either archimedean or for which $b_1, \ldots, b_n$ are not all integral over $\mathcal{O}_{K_v}$ or for which the discriminant $\mathrm{disc}(b_1, \ldots, b_n)$ is not a unit in $\mathcal{O}_{K_v}$. For every $v \in M_K \smallsetminus S_1$ the maps (1) and Proposition 9.5.6 then induce isomorphisms

$$\mathcal{O}_{K_v}^n \xrightarrow{\sim} \mathcal{O}_{K_v} \otimes_{\mathcal{O}_K} \mathcal{O}_L \xrightarrow{\sim} \times_{w|v} \mathcal{O}_{L_w}. \tag{3}$$

Thus an element $(a_{v,i})_{v,i} \in \times_{v \in M_K} K_v^n$ satisfies the additional condition for $\mathbb{A}_K^n$ if and only if its image under the isomorphisms (2) satisfies the additional condition for $\mathbb{A}_L$. The maps (2) therefore induce isomorphisms

$$\mathbb{A}_K^n \xrightarrow{\sim} \mathbb{A}_K \otimes_K L \xrightarrow{\sim} \mathbb{A}_L. \tag{4}$$

Here the second isomorphism is natural, because it is independent of the choice of basis.

Now consider an arbitrary finite subset $S \subset M_K$ containing S_1, and let T be the set of places of L above all places in S. Then the above isomorphisms reduce to isomorphisms

$$\begin{array}{ccc} (\mathbb{A}_K^S)^n & \xrightarrow{\quad\sim\quad} & \mathbb{A}_L^T \\ \| & & \| \\ \bigtimes_{v\in S} K_v^n \times \bigtimes_{v\in M_K\smallsetminus S} \mathcal{O}_{K,v}^n & \xrightarrow{\ \sim\ } & \bigtimes_{w\in T} L_w \times \bigtimes_{w\in M_L\smallsetminus T} \mathcal{O}_{K,v}. \end{array} \tag{5}$$

Since $\mathbb{A}_K^S$ and $\mathbb{A}_L^T$ carry the product topology and (1) and (3) are topological isomorphisms, it follows that (5) is a topological isomorphism. Finally, as S varies, the sets T are cofinal among all finite subsets of M_L. Taking the union we conclude that (4) is a topological isomorphism, as desired.

(b) By assumption K is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for a prime p. By (a) it therefore suffices to prove the assertion for $K = \mathbb{Q}$ and $K = \mathbb{F}_p(t)$.

In the case $K = \mathbb{Q}$ set $A := \mathbb{Z}$ and let ∞ denote the archimedean place. In the case $K = \mathbb{F}_p(t)$ set $A := \mathbb{F}_p[t]$ and let ∞ denote the place with the normalized valuation $v_\infty(f/g) = \deg(f) - \deg(g)$. In either case A is a principal ideal domain and the places $v \neq \infty$ of K are in bijection with the equivalence classes of prime elements p of A. In particular we have

$$\mathbb{A}_K^{\{\infty\}} \;=\; K_\infty \times \bigtimes_{p\neq\infty} A_p \tag{6}$$

and

$$\mathbb{A}_K^{\{\infty\}} \cap K \;=\; A. \tag{7}$$

We also claim that

$$\mathbb{A}_K^{\{\infty\}} + K \;=\; \mathbb{A}_K. \tag{8}$$

To see this take any $\underline{a} = (a_v)_v \in \mathbb{A}_K$ and choose a finite subset $S \subset M_K$ with $\underline{a} \in \mathbb{A}_K^S$. Then for any $p \in S \smallsetminus \{\infty\}$ we have $K_p = A_p[p^{-1}] = A_p + A[p^{-1}]$, so we can choose an element $b_p \in A[p^{-1}]$ with $a_p \in A_p + b_p$. Since this b_p is integral outside p, with $b := \sum_p b_p$ we then have $a_p \in A_p + b$ for all $p \neq \infty$. Altogether we deduce that $\underline{a} \in \mathbb{A}_K^{\{\infty\}} + b \subset \mathbb{A}_K^{\{\infty\}} + K$, as desired.

Now recall that by definition $\mathbb{A}_K^{\{\infty\}}$ is an open subgroup of $\mathbb{A}_K$. Thus K is discrete in $\mathbb{A}_K$ if and only if $\mathbb{A}_K^{\{\infty\}} \cap K$ is discrete in $\mathbb{A}_K^{\{\infty\}}$. Since equations (7) and (8) yield a topological isomorphism $\mathbb{A}_K^{\{\infty\}}/A \cong \mathbb{A}_K/K$, it suffices to prove that A is discrete and cocompact in $\mathbb{A}_K^{\{\infty\}}$.

Since $\mathbb{A}_K^{\{\infty\}}$ carries the product topology in (6) and all factors A_p are compact, it suffices to show that the image of A in K_∞ is discrete and cocompact. For $\mathbb{Z} \subset \mathbb{R}$ this is of course well-known. In the case $K = \mathbb{F}_p(t)$ we observe that

$$K_\infty \;\cong\; \mathbb{F}_p((t^{-1})) \;=\; \mathbb{F}_p[t] \oplus \mathbb{F}_p[[t^{-1}]] \cdot t^{-1},$$

where the second factor is open and compact. Thus $A = \mathbb{F}_p[t]$ is discrete in K_∞ and the quotient $K_\infty/A \cong \mathbb{F}_p[[t^{-1}]] \cdot t^{-1}$ is compact, as desired.

(c) See Section 25.4 in
`https://math.mit.edu/classes/18.785/2017fa/LectureNotes25.pdf`.

(d) The answer is "No". We only explain the number field case, the function field case being similar. Suppose that $K^\times \cdot K_v^\times$ dense in I_K. Let S be union of $\{v\}$ with the set of all archimedean places of K. Then

$$(K^\times \cdot K_v^\times) \cap I_K^S \;=\; (K^\times \cap I_K^S) \cdot K_v^\times$$

must be dense in the open subgroup I_K^S of I_K.

Claim: The group $K^\times \cap I_K^S$ is finitely generated.

Proof: By definition of I_K^S the group $G := K^\times \cap I_K^S$ consists of all $a \in K^\times$ that are units at all non-archimedean places $\neq v$ of K. If v is archimedean, this is simply the group of units $\mathcal{O}_K$ and therefore finitely generated by Dirichlet's unit theorem. If v is non-archimedean, the image of the associated normalized valuation $v\colon G \to \mathbb{Z}$ is of course finitely generated, so it suffices to show that the kernel is finitely generated. But in the number field case this kernel is again just $\mathcal{O}_K^\times$ and therefore finitely generated. $\square$

Now recall that

$$I_K^S \;=\; \bigtimes_{v\in S} K_v^\times \times \bigtimes_{v\in M_K\smallsetminus S} \mathcal{O}_{K_v}^\times$$

with the product topology. Here $\mathcal{O}_{K_v}^\times$ is a profinite abelian group with a finite cyclic direct factor μ_{K_v} of even order, so it possesses a continuous surjective homomorphism $\mathcal{O}_{K_v}^\times \twoheadrightarrow \mathbb{F}_2$. Together this yields a continuous surjective homomorphism $I_K^S \twoheadrightarrow \mathbb{F}_2^{M_K\smallsetminus S}$. (Compare Exercise 1 from Series 24). But since $M_K \smallsetminus S$ is infinite, the image of a finitely generated subgroup can never be dense in $\mathbb{F}_2^{M_K\smallsetminus S}$. Thus $K^\times \cap I_K^S$ cannot be dense in I_K^S, and we have a contradiction.

3. Let K be a finite extension of $\mathbb{F}_p(t)$. Let M_K denote the set of normalized valuations on K and let k_v denote the residue field at $v \in M_K$. Prove the *product formula* for all $x \in K^\times$:

$$\prod_{v\in M_K} |k_v|^{-v(x)} \;=\; 1.$$

Solution For the field $K_0 := \mathbb{F}_p(t)$ we know this from exercise 1 of sheet 16. For a finite extension K/K_0 we know from the lecture that for every $v_0 \in M_{K_0}$ we have

$$\prod_{\substack{v\in M_K\\ v|v_0}} |k_v|^{-v(x)} \;=\; |k_{v_0}|^{-v_0(\mathrm{Nm}_{K/K_0}(x))}.$$

Thus the product formula for K follows directly from that for K_0.