D-MATH | Number Theory II | FS 2024
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Exercise sheet 27

Class Fields, Reciprocity Laws

1. Let K be a number field. Call an element $x \in K^\times$ *totally positive* if it becomes positive under every real embedding of K. Let $\mathrm{Cl}'(\mathcal{O}_K)$ denote the group of all fractional ideals of $\mathcal{O}_K$ modulo the subgroup of principal ideals generated by totally positive elements of $K^\times$. Show that the maximal abelian extension H/K that is everywhere unramified possesses a natural isomorphism

$$\mathrm{Gal}(H/K) \;\cong\; \mathrm{Cl}'(\mathcal{O}_K).$$

2. Deduce the two supplements of the quadratic reciprocity law from the reciprocity isomorphism of global class field theory.

3. *(A cubic reciprocity law)* Recall that the number field $K := \mathbb{Q}(\mu_3) = \mathbb{Q}(\sqrt{-3})$ is imaginary quadratic, that $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is a principal ideal domain, and that $3\mathcal{O}_K = \mathfrak{m}^2$ for the maximal ideal $\mathfrak{m} := (\sqrt{-3})$. Take inequivalent primes $\pi, \rho \in \mathcal{O}_K \smallsetminus \mathfrak{m}$ and consider the extension $L := K(\sqrt[3]{\pi})$ of K, which by Kummer theory is cyclic with Galois group μ_3.

   (a) Show that all primes $\neq \mathfrak{m}, (\pi)$ of $\mathcal{O}_K$ are unramified in L.

   (b) Show that $\mathfrak{m}$ is unramified in L if and only if $\pi \equiv \pm 1 \bmod \mathfrak{m}^3$.

   (c) Assuming this, prove that the residue class of π is a cube in the residue field $\mathcal{O}_K/(\rho)$ if and only if the residue class of ρ is a cube in $\mathcal{O}_K/(\pi)$.