8.3 Complete valuations

Definition 8.3.1: The *completion* of a ring A with respect to an ideal $\mathfrak{a}$ is the subring

$$A_{\mathfrak{a}} := \varprojlim_k (A/\mathfrak{a}^k) := \Big\{(x_k + \mathfrak{a}^k)_k \in \bigtimes_{k\geqslant 0} (A/\mathfrak{a}^k) \;\Big|\; \forall k \geqslant 0\colon x_k \equiv x_{k+1} \bmod \mathfrak{a}^k\Big\}.$$

It is equipped with a natural ring homomorphism

$$\begin{tikzcd} \cdots \arrow[r, two heads] & A/\mathfrak{a}^{k+2} \arrow[r, two heads] & A/\mathfrak{a}^{k+1} \arrow[r, two heads] & A/\mathfrak{a}^{k} \\ & & A \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$i\colon A \longrightarrow A_{\mathfrak{a}},\; x \mapsto (x+\mathfrak{a}^k)_k.$$

Example 8.3.2: For any ring R the completion of $R[t]$ with respect to the ideal (t) is naturally isomorphic to $R[[t]]$.

$A \qquad \mathfrak{a}$

$$R[t]/(t)^k = \{ a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + (t^k) \mid a_i \in R \text{ unique} \}.$$

$$R[[t]] = \{ a_0 + a_1 t + \ldots \mid \text{all } a_i \in R \}.$$

Example 8.3.3: The ring $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the ideal (p).

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

Now consider a Dedekind ring A with quotient field K and a maximal ideal $\mathfrak{p}$. Pick an element $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$.

Proposition 8.3.4: (a) The set of units of $A_{\mathfrak{p}}$ is $\boxed{A_{\mathfrak{p}}^{\times} = A_{\mathfrak{p}} \smallsetminus \pi A_{\mathfrak{p}}.}$

$A_{\mathfrak{p}}^{\times} = \varprojlim (A/\mathfrak{p}^k)^{\times}$

(b) The ideal $\mathfrak{m}_{\mathfrak{p}} := (\pi)$ of $A_{\mathfrak{p}}$ is the unique maximal ideal.

(c) Every nonzero ideal of $A_{\mathfrak{p}}$ is equal to $\mathfrak{m}_{\mathfrak{p}}^r$ for a unique integer $r \geqslant 0$.

(d) The ring $A_{\mathfrak{p}}$ is a principal ideal domain. $\Rightarrow$ Dedekind.

(e) It is the valuation ring for the discrete valuation $\mathrm{ord}_{\mathfrak{m}_{\mathfrak{p}}}$ on the field $K_{\mathfrak{p}} := A_{\mathfrak{p}}[\pi^{-1}]$.

(f) The natural homomorphism $i\colon A \to A_{\mathfrak{p}}$ is injective.

(g) It therefore induces an injective homomorphism $i\colon K \hookrightarrow K_{\mathfrak{p}}$.

(h) For any $x \in K$ we have $\mathrm{ord}_{\mathfrak{p}}(x) = \mathrm{ord}_{\mathfrak{m}_{\mathfrak{p}}}(i(x))$.

Proof: (a): $\forall k \geq 1$: $(A/\mathfrak{p}^k)^{\times} = \{a + \mathfrak{p}^k \mid a \in A \smallsetminus \mathfrak{p}\}$

$= \{a + \mathfrak{p}^k \mid a \in A \smallsetminus \pi A\}$

$(a + \mathfrak{p}^k) = (1)$ in $A/\mathfrak{p}^k$

$\Leftrightarrow \underbrace{(a) + \mathfrak{p}^k}_{= \mathfrak{p}^\ell \text{ for some } \ell \leq k} = (1) = A$

$\Leftrightarrow \ell = 0$

$\Leftrightarrow a \notin \mathfrak{p}$.

$a \in \pi A \Rightarrow a \in \mathfrak{p}$

Claim: $\underbrace{(\pi) + \mathfrak{p}^k}_{= \mathfrak{p}^\ell \text{ for some } \ell \leq k} \overset{!}{=} \mathfrak{p}$.

$\pi \in \mathfrak{p} \Rightarrow \ell \geq 1$.

$\pi \notin \mathfrak{p}^2 \Rightarrow \ell < 2$

$(A/\mathfrak{p}^k) \smallsetminus (A/\mathfrak{p}^k)^{\times} = \pi \cdot A/\mathfrak{p}^k$

$\varprojlim$: $A_{\mathfrak{p}} \smallsetminus A_{\mathfrak{p}}^{\times} = \pi \cdot A_{\mathfrak{p}}$.

(b) $\underset{A_g}{\boxed{1}} = \big(\underbrace{1+g^k}_{\neq 0 \text{ for } k>0}\big)_k \boxed{\neq 0}$ and $1 \notin \pi\cdot A_g$. So $\pi A_g \subsetneq A_g$.

Since $A_g \setminus \pi A_g \subseteq A_g^\times \Rightarrow \pi A_g$ is max. ideal.

For any ideal $I \subsetneq A_g$ we have $I \cap A_g^\times = \emptyset$. So (a) $\Rightarrow I \subset \pi A_g$.

(c) $I \subset A_g$ nonzero; $\underline{x} = (x_k + g^k)_k \in I \setminus \{0\} \Rightarrow \exists$ minimal k with $x_{k+1}+g^{k+1} \neq g^{k+1}$

So $\underline{x} \notin \pi^{k+1} A_g$.

So $\exists!\, k \geq 0$ with $I \subset \pi^k A_g$ and $I \not\subset \pi^{k+1} A_g$.

Any $x \in I \setminus \pi^{k+1} A_g$ has the form $\pi^k \cdot y$ for $y \in A_g \setminus \pi A_g = A_g^\times$

$\Rightarrow \pi^k A_g = y \cdot A_g \subset I$. $\Rightarrow I = \pi^k A_g$.

Also $\pi^k A_g \supsetneq \pi^{k+1} A_g$

(d) $\forall x, y \in A_g \setminus \{0\}$: $\left.\begin{array}{l} x \in \pi^k \cdot A_g^\times, \\ y \in \pi^l \cdot A_g^\times \end{array}\right\} \Rightarrow xy \in \pi^{k+l} \cdot A_g^\times \not\ni 0$.

(e) By (d): $\mathrm{Quot}(A_g) = A_g[\frac{1}{\pi}] = \bigcup_{n\geq 0} \frac{1}{\pi^n} \cdot A_g$.

$K_g = \{0\} \cup \coprod_{n \in \mathbb{Z}} \pi^n \cdot A_g^\times$

(f) Let $a \in \ker(A \to A_{\mathfrak{p}})$.

Then $\forall k \geq 0$: $a + \mathfrak{p}^k = \mathfrak{p}^k$, i.e. $a \in \mathfrak{p}^k$.

$\Rightarrow a \in \bigcap_{n \geq 0} \mathfrak{p}^n$.

If $a \neq 0$ then $(a) = \mathfrak{p}^\ell$. (Π other prin ideals) $\Rightarrow (a) \not\subseteq \mathfrak{p}^{\ell+1}$.

$\ell = \operatorname{ord}_{\mathfrak{p}}(a)$

So $\bigcap_{n \geq 0} \mathfrak{p}^n = (0)$.

(g) ✓

(h)

$a + \mathfrak{p}^k \in (\pi^\ell) + \mathfrak{p}^k$ for all $k \geq 0$

$\Rightarrow i(a) \in \pi^\ell A_{\mathfrak{p}}$.

$a + \mathfrak{p}^k \notin (\pi^{\ell+1}) + \mathfrak{p}^k$ if $k \gg 0$.

$\Rightarrow i(a) \notin \pi^{\ell+1} A_{\mathfrak{p}}$

$\Rightarrow \operatorname{ord}_{\mathfrak{m}_{\mathfrak{p}}}(i(a)) = \ell = \operatorname{ord}_{\mathfrak{p}}(a)$.

qed