Reminder:

Consider a Dedekind ring A with quotient field K and a maximal ideal \mathfrak{p} . Pick an element $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$.

Definition 8.3.1: The *completion* of A with respect to \mathfrak{p} is the subring

$$A_{\mathfrak{p}} := \lim_{\stackrel{\leftarrow}{k}} (A/\mathfrak{p}^k) := \Big\{ (x_k + \mathfrak{p}^k)_k \in \bigotimes_{k \ge 0} (A/\mathfrak{p}^k) \ \Big| \ \forall k \ge 0 \colon x_k \equiv x_{k+1} \bmod \mathfrak{p}^k \Big\}.$$

It is equipped with a natural ring homomorphism

$$i: A \longrightarrow A_{\mathfrak{p}}, \ x \mapsto (x + \mathfrak{p}^k)_k.$$

Proposition 8.3.4: (a) The set of units of $A_{\mathfrak{p}}$ is $A_{\mathfrak{p}}^{\times} = A_{\mathfrak{p}} \smallsetminus \pi A_{\mathfrak{p}}$. (b) The ideal $\mathfrak{m}_{\mathfrak{p}} := (\pi)$ of $A_{\mathfrak{p}}$ is the unique maximal ideal. (c) Every nonzero ideal of $A_{\mathfrak{p}}$ is equal to $\mathfrak{m}_{\mathfrak{p}}^r$ for a unique integer $r \ge 0$. (d) The ring $A_{\mathfrak{p}}$ is a principal ideal domain. \Longrightarrow Dedektion . (e) It is the valuation ring for the discrete valuation $\operatorname{ord}_{\mathfrak{m}_{\mathfrak{p}}}$ on the field $K_{\mathfrak{p}} := A_{\mathfrak{p}}[\pi^{-1}]$. (f) The natural homomorphism $i: A \to A_{\mathfrak{p}}$ is injective. (g) It therefore induces an injective homomorphism $i: K \hookrightarrow K_{\mathfrak{p}}$.

(h) For any $x \in K$ we have $\operatorname{ord}_{\mathfrak{p}}(x) = \operatorname{ord}_{\mathfrak{m}_{\mathfrak{p}}}(i(x))$.

Definition 8.3.5: A normalized discrete valuation v on a field K is called *complete* if the natural homomorphism $i: K \hookrightarrow K_{\mathfrak{m}_v}$ is an isomorphism.

Proposition 8.3.6: The valuation $\operatorname{ord}_{\mathfrak{m}_p}$ on the completion K_p is complete.

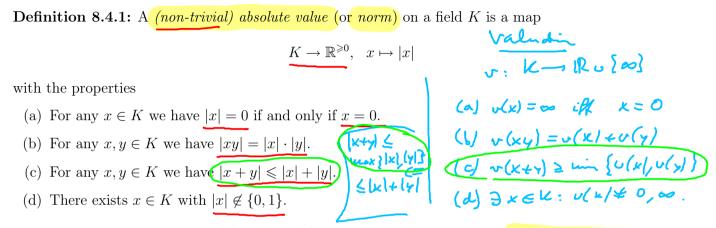
$$\frac{(4m)!}{1m} = \frac{1}{2} \frac{1}{1m} = \frac{1}{2} \frac{1}{2} \frac{1}{1m} = \frac{1}{2} \frac{1}{2}$$

Example 8.3.7: The valuations on k((t)) and on \mathbb{Q}_p are complete.

$$u[t]_{(t)} = u[[t]]$$

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8.4 Absolute Values



Remark 8.4.2: The map with |0| = 0 and |x| = 1 for all $x \neq 0$ is called the *trivial absolute value*. Some of the results below also hold for it, and sometimes one allows it as well, but we exclude it without further mention.

Fact: For any valuation v on K and any constant 0 < c < 1 the map $|x| := c^{v(x)}$ is an absolute value on K.

Caution 8.4.3: Don't confuse an absolute value with a valuation, as many do. \odot

Example 8.4.4: The usual absolute value on \mathbb{R} or \mathbb{C} or any subfield thereof.

Proposition 8.4.5: For any absolute value | | and any real number $0 < s \leq 1$ the map $| |^s$ is also an absolute value.

Proposition 8.4.6: Any absolute value | | on a field K turns K into a metric space with the metric d(x, y) := |x - y|.

Proposition-Definition 8.4.7: For any two absolute values | | and | |' on K the following are equivalent:

(a) They define the same topology on K.

(b) For any $x \in K$ we have |x|' < 1 if and only if |x| < 1.

(c) There exists a real number s > 0 such that for all $x \in K$ we have $|x|' = |x|^s$.

Two such absolute values are called *equivalent*.

$$\begin{array}{c} \begin{array}{c} \left[\begin{array}{c} u \\ u \\ \end{array} \right]_{k} \left(c \right) = 1 \left(c \right) \\ \left(c \right) = 1 \left(c \right) \\ \left(c$$

(1) | x / y / < 7 = ... > | x | < |y| m/

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Definition 8.4.8: An absolute value | | is called *ultrametric* if it satisfies the stronger property

(c') For any $x, y \in K$ we have $|x + y| \leq \max\{|x|, |y|\}$. $\sim |x_1 + \ldots + |x_n| \leq \max\{|x_1|, \ldots, |x_n|\}$

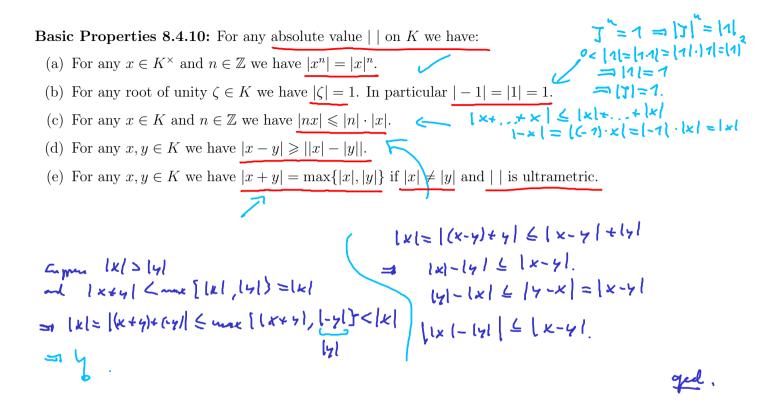
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ACOD - UK: V(K)= S. V(x) (=)

Proposition 8.4.9: (a) For any valuation v on K and any constant 0 < c < 1 the map $|x| := c^{v(x)}$ is an ultrametric absolute value on K.

 $c^{\nu'(k)} = \left(c^{\nu(k)}\right)^{s}$

- (b) Any ultrametric absolute value arises in this fashion from a valuation. \checkmark
- (c) Two valuations are equivalent if and only if the associated absolute values are equivalent.



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Definition 8.4.11: An absolute value || is called *archimedean* if for every $x \in K$ there exists $n \in \mathbb{Z}$ with $|x| \leq n$. Otherwise it is called *nonarchimedean*.

Theorem 8.4.12: An absolute value | | is ultrametric if and only if it is nonarchimedean.

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$$\frac{\int u_{n} d_{n}}{\left(1 - \frac{1}{2} -$$

Definition 8.4.13: On the field \mathbb{Q} we have the usual absolute value

$$|x|_{\infty} := \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0, \end{cases}$$

and for every prime number p the *p*-adic absolute value

$$|x|_p := \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

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where $\operatorname{ord}_p(x)$ is the exponent of p in the prime factorization of x.

The normalization in Definition 8.4.13 is specially chosen to achieve:

Theorem 8.4.14: For every $x \in \mathbb{Q}^{\times}$ we have

$$\begin{split} \prod_{p \leq \infty} |x|_p &= 1. \\ \hline \prod_{p \leq \infty} |x|_p &= 1. \\ \hline p \leq \infty \\ \hline$$

Theorem 8.4.15: (Ostrowski) Every absolute value on \mathbb{Q} is equivalent to exactly one of the above. **Proof:** Consider an arbitrary absolute value $\| \| \|$ on \mathbb{Q} . If it is archimedean, then: Claim 1: $\forall m, n \in \mathbb{Z}^{>1}$: $||m|| \leq \left(\frac{\log m}{\log n} + 1\right) \cdot n \cdot ||n||^{\frac{\log m}{\log n}}$. $\|u\| \ge 1^{2}$ Claim 2: $\forall m, n \in \mathbb{Z}^{>1}$: $||m|| \leq ||n||^{\frac{\log m}{\log n}}$. Claim 3: $\forall m, n \in \mathbb{Z}^{>1}$: $||m||^{\frac{1}{\log m}} = ||n||^{\frac{1}{\log n}}$. Unik $m = \sum_{i=0}^{k} a_i n^i$ with $a_i \in \{0, -7^{k-7}\}$ $\Rightarrow \|\|\mathbf{u}\| \leq \sum_{i=0}^{k} \|\mathbf{u}_{i}\| \|\mathbf{u}\|^{i} \leq \mathbf{u} \cdot (k+1) \cdot \max\{1, \|\mathbf{u}\|^{k}\} \leq \mathbf{u} \cdot (\frac{k_{0}}{k_{1}} + 1) \cdot \max\{1, \|\mathbf{u}\|\}$ n E m K n K +1 = kelyn Elaym E (k+1) lag n L' Lyn $Close 1 \rightarrow \forall l \ge 1: || m ||^{l} = || m^{l} || \leq \left(\frac{l \cdot l \cdot m}{l \cdot m} + 1 \right) \cdot h \cdot m \cdot m \cdot k \leq || n || \frac{l \cdot m}{l \cdot m}, 1 \geq l$ $= \lim_{k \to \infty} \frac{1}{k} = 2 \cdot \log n$ $= \lim_{k \to \infty} \frac{1}{k} \cdot \log \left(2 \cdot \frac{\log n}{\log n} + 1 \right) \cdot n \cdot \max \left\{ \left\| n \right\|^{\frac{1}{2}} + \frac{1}{2} \right\}$