Reminder:

Consider a Dedekind ring A with quotient field K and a maximal ideal $\mathfrak{p}$. Pick an element $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$.

Definition 8.3.1: The *completion* of A with respect to $\mathfrak{p}$ is the subring

$$A_\mathfrak{p} \ := \ \varprojlim_k (A/\mathfrak{p}^k) \ := \ \Big\{ (x_k + \mathfrak{p}^k)_k \in \bigtimes_{k \geqslant 0} (A/\mathfrak{p}^k) \ \Big| \ \forall k \geqslant 0 \colon x_k \equiv x_{k+1} \bmod \mathfrak{p}^k \Big\}.$$

It is equipped with a natural ring homomorphism

$$i \colon A \longrightarrow A_\mathfrak{p}, \ \ x \mapsto (x + \mathfrak{p}^k)_k.$$

$\mathbb{Z} \hookrightarrow \mathbb{Z}_p$

Proposition 8.3.4: (a) The set of units of $A_\mathfrak{p}$ is $A_\mathfrak{p}^\times = A_\mathfrak{p} \smallsetminus \pi A_\mathfrak{p}$.

(b) The ideal $\mathfrak{m}_\mathfrak{p} := (\pi)$ of $A_\mathfrak{p}$ is the unique maximal ideal. $\mathfrak{m}_\mathfrak{p}$

(c) Every nonzero ideal of $A_\mathfrak{p}$ is equal to $\mathfrak{m}_\mathfrak{p}^r$ for a unique integer $r \geqslant 0$.

(d) The ring $A_\mathfrak{p}$ is a principal ideal domain. $\Rightarrow$ Dedekind. $A_\mathfrak{p}$

(e) It is the valuation ring for the discrete valuation $\mathrm{ord}_{\mathfrak{m}_\mathfrak{p}}$ on the field $K_\mathfrak{p} := A_\mathfrak{p}[\pi^{-1}]$.

(f) The natural homomorphism $i \colon A \to A_\mathfrak{p}$ is injective.

(g) It therefore induces an injective homomorphism $i \colon K \hookrightarrow K_\mathfrak{p}$.

(h) For any $x \in K$ we have $\mathrm{ord}_\mathfrak{p}(x) = \mathrm{ord}_{\mathfrak{m}_\mathfrak{p}}(i(x))$.

Definition 8.3.5: A normalized discrete valuation v on a field K is called *complete* if the natural homomorphism $i\colon K \hookrightarrow K_{\mathfrak{m}_v}$ is an isomorphism.

Proposition 8.3.6: The valuation $\mathrm{ord}_{\mathfrak{m}_\mathfrak{p}}$ on the completion $K_\mathfrak{p}$ is complete.

Proof: To prove: $A_\mathfrak{p} \longrightarrow (A_\mathfrak{p})_{\mathfrak{m}_\mathfrak{p}}$ is an isom.

$$A_\mathfrak{p} = \varprojlim_n (A/\mathfrak{p}^n), \qquad (A_\mathfrak{p})_{\mathfrak{m}_\mathfrak{p}} = \varprojlim_n (A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}^n)$$

$\rightsquigarrow$ Enough: $\forall n$: $A/\mathfrak{p}^n \longrightarrow A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}^n$ is an isom.

$$A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}^n = A_\mathfrak{p}/\pi^n A_\mathfrak{p}$$

Claim: $A_\mathfrak{p} \longrightarrow A/\mathfrak{p}^n$ surjective

For any $a \in A$: $a+\mathfrak{p}^n$

$(a+\mathfrak{p}^k)_k \longmapsto a+\mathfrak{p}^n$.

$n=1$: $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \cong A/\mathfrak{p}$.

n arbitrary: Each side is an extension of n copies thereof.

qed.

Example 8.3.7: The valuations on $k((t))$ and on $\mathbb{Q}_p$ are complete.

$$k[t]_{(t)} = k[[t]]$$

8.4 Absolute Values

Definition 8.4.1: A *(non-trivial) absolute value* (or *norm*) on a field K is a map

$$K \to \mathbb{R}^{\geqslant 0}, \quad x \mapsto |x|$$

with the properties

(a) For any $x \in K$ we have $|x| = 0$ if and only if $x = 0$.

(b) For any $x, y \in K$ we have $|xy| = |x| \cdot |y|$.

(c) For any $x, y \in K$ we have $|x + y| \leqslant |x| + |y|$.

(d) There exists $x \in K$ with $|x| \notin \{0, 1\}$.

$|x+y| \leq \max\{|x|,|y|\} \leq |x|+|y|$

Valuation

$v : K \to \mathbb{R} \cup \{\infty\}$

(a) $v(x) = \infty$ iff $x = 0$

(b) $v(xy) = v(x) + v(y)$

(c) $v(x+y) \geq \min\{v(x), v(y)\}$

(d) $\exists x \in K : v(x) \neq 0, \infty$.

Remark 8.4.2: The map with $|0| = 0$ and $|x| = 1$ for all $x \neq 0$ is called the *trivial absolute value.* Some of the results below also hold for it, and sometimes one allows it as well, but we exclude it without further mention.

Fact: For any valuation v on K and any constant $0 < c < 1$ the map $|x| := c^{v(x)}$ is an absolute value on K.

Caution 8.4.3: Don't confuse an absolute value with a valuation, as many do. ☹

Example 8.4.4: The usual absolute value on $\mathbb{R}$ or $\mathbb{C}$ or any subfield thereof.

Proposition 8.4.5: For any absolute value $|\ |$ and any real number $0 < s \leqslant 1$ the map $|\ |^s$ is also an absolute value.

Proposition 8.4.6: Any absolute value $|\ |$ on a field K turns K into a metric space with the metric $d(x,y) := |x-y|$.

Proposition-Definition 8.4.7: For any two absolute values $|\ |$ and $|\ |'$ on K the following are equivalent:

(a) They define the same topology on K.

(b) For any $x \in K$ we have $|x|' < 1$ if and only if $|x| < 1$.

(c) There exists a real number $s > 0$ such that for all $x \in K$ we have $|x|' = |x|^s$.

Two such absolute values are called *equivalent.*

Proof: (c) $\Rightarrow$ (a) ✓

(a) $\Rightarrow$ (b): $|x| < 1 \iff |x^n| = |x|^n \to 0$ for $n \to \infty$.

$\overset{(a)}{\iff} |x^n|' = |x|'^n \to 0$ " . . . . $\iff |x|' < 1$.

(b) $\Rightarrow$ (c): Fix $y \in K$ with $|y| \neq 0, 1$. WLOG: $|y| > 1$.

Take any $x \in K$. Then $|x| = |y|^\alpha$ for unique $\alpha \in \mathbb{R}$.

For any $m, n \in \mathbb{Z}$, $n > 0$: $\frac{m}{n} > \alpha \Rightarrow |x| < |y|^{m/n} \Rightarrow |x^n| = |x|^n < |y|^m = |y^m|$.

$\Rightarrow |x^n/y^m| = \frac{|x^n|}{|y^m|} < 1$.

$\overset{(b)}{\Rightarrow} \left| x^n / y^m \right|' < 1 \Rightarrow \dots \Rightarrow |x|' < |y|'^{\,m/n}$

$\forall$ any $\frac{m}{n} \Rightarrow |x|' \le |y|'^{\,\alpha}$

Repeat with $\frac{m}{n} < \alpha \Rightarrow \dots \Rightarrow |x|' \ge |y|'^{\,\alpha}$

$\left.\begin{array}{l} |x|' \le |y|'^{\,\alpha} \\ |x|' \ge |y|'^{\,\alpha} \end{array}\right\} \Rightarrow |x|' = |y|'^{\,\alpha}$

$\Rightarrow \left.\begin{array}{l} \log|x|' = \alpha \cdot \log|y|' = \log|x| \cdot \underbrace{\dfrac{\log|y|'}{\log|y|}}_{s} \\ \log|x| = \alpha \cdot \log|y| \end{array}\right\} \Rightarrow \begin{array}{l} \log|x|' = \log|x| \cdot s \\ \Rightarrow |x|' = |x|^s . \end{array}$

qed.

$|nx| \leq |x|.$

Definition 8.4.8: An absolute value $|\ |$ is called *ultrametric* if it satisfies the stronger property

(c′) For any $x, y \in K$ we have $|x + y| \leqslant \max\{|x|, |y|\}$.

$\rightsquigarrow |x_1 + \ldots + x_n| \leq \max\{|x_1|, \ldots, |x_n|\}$

Proposition 8.4.9: (a) For any valuation v on K and any constant $0 < c < 1$ the map $|x| := c^{v(x)}$ is an ultrametric absolute value on K.

(b) Any ultrametric absolute value arises in this fashion from a valuation.

(c) Two valuations are equivalent if and only if the associated absolute values are equivalent.

$\Updownarrow$

$\exists s > 0 : \forall x : v'(x) = s \cdot v(x) \iff c^{v'(x)} = \left(c^{v(x)}\right)^s$

Basic Properties 8.4.10: For any absolute value $|\ |$ on K we have:

(a) For any $x \in K^\times$ and $n \in \mathbb{Z}$ we have $|x^n| = |x|^n$.

(b) For any root of unity $\zeta \in K$ we have $|\zeta| = 1$. In particular $|-1| = |1| = 1$.

(c) For any $x \in K$ and $n \in \mathbb{Z}$ we have $|nx| \leqslant |n| \cdot |x|$.

(d) For any $x, y \in K$ we have $|x - y| \geqslant ||x| - |y||$.

(e) For any $x, y \in K$ we have $|x + y| = \max\{|x|, |y|\}$ if $|x| \neq |y|$ and $|\ |$ is ultrametric.

$\zeta^n = 1 \Rightarrow |\zeta|^n = |1|$

$0 < |1| = |1 \cdot 1| = |1| \cdot |1| = |1|^2$

$\Rightarrow |1| = 1$

$\Rightarrow |\zeta| = 1.$

$|x + \dots + x| \leq |x| + \dots + |x|$

$|-x| = |(-1) \cdot x| = |-1| \cdot |x| = |x|$

$|x| = |(x-y) + y| \leq |x-y| + |y|$

$\Rightarrow |x| - |y| \leq |x-y|.$

$|y| - |x| \leq |y-x| = |x-y|$

$||x| - |y|| \leq |x-y|.$

Suppose $|x| > |y|$

and $|x+y| < \max\{|x|, |y|\} = |x|$

$\Rightarrow |x| = |(x+y) + (-y)| \leq \max\{|x+y|, \underbrace{|-y|}_{|y|}\} < |x|$

$\Rightarrow \lightning$.

qed.

Definition 8.4.11: An absolute value $|\ |$ is called *archimedean* if ~~for every $x \in K$ there exists $n \in \mathbb{Z}$ with $|x| < n$~~ $\forall c \in \mathbb{R}\ \exists n \in \mathbb{Z} : |n \cdot 1| > c.$ Otherwise it is called *nonarchimedean.*

Theorem 8.4.12: An absolute value $|\ |$ is ultrametric if and only if it is nonarchimedean.

Proof: $|\cdot|$ ultrametric $\Rightarrow \forall n \in \mathbb{Z}: \ |n \cdot 1| = \begin{cases} n \geq 0 \Rightarrow \ \leq \max\{|1|, \dots, |1|\} = 1. \\ n < 0 \Rightarrow \ = |-n \cdot 1| \leq 1. \end{cases}$

$\Rightarrow |\cdot|$ nonarchimedean.

$|\cdot|$ nonarchimedean, i.e. $\exists c > 0 : \forall n \in \mathbb{Z}: |n \cdot 1| \leq c.$

Take $x, y \in K$ with $|x| \geq |y|$.

$$\Rightarrow |x+y|^n = |(x+y)^n| = \Big|\sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu y^{n-\nu}\Big| \leq \sum_{\nu=0}^{n} \underbrace{\Big|\binom{n}{\nu} \cdot 1\Big|}_{\leq c} \cdot \underbrace{|x|^\nu \cdot |y|^{n-\nu}}_{\leq |x|^n}$$

$$\leq (n+1) \cdot c \cdot |x|^n$$

$$\Rightarrow |x+y| \leq \underbrace{\sqrt[n]{(n+1) \cdot c}}_{\to 1 \text{ for } n \to \infty} \cdot |x|. \quad \Rightarrow |x+y| \leq |x|.$$

qed.

Definition 8.4.13: On the field $\mathbb{Q}$ we have the usual absolute value

$$\boxed{|x|_\infty := \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0, \end{cases}}$$

archimedean

and for every prime number p the *p-adic absolute value*

$$\boxed{|x|_p := \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}}$$

$(p^{-1})^{\operatorname{ord}_p(x)}$

non-archimedean.

where $\operatorname{ord}_p(x)$ is the exponent of p in the prime factorization of x.

The normalization in Definition 8.4.13 is specially chosen to achieve:

Theorem 8.4.14: For every $x \in \mathbb{Q}^\times$ we have

$$\boxed{\prod_{p \leqslant \infty} |x|_p = 1.}$$

Proof: $x = \pm \prod_p' p^{\nu_p}$ ← $\in \mathbb{Z}$, almost all $=0$.

p ← prime

$$|x|_\infty = \prod_p p^{\nu_p}$$

$$\operatorname{ord}_p(x) = \nu_p$$

$$|x|_p = p^{-\nu_p}$$

qed.

Theorem 8.4.15: *(Ostrowski)* Every absolute value on $\mathbb{Q}$ is equivalent to exactly one of the above.

Proof: Consider an arbitrary absolute value $\| \ \|$ on $\mathbb{Q}$. If it is archimedean, then:

Claim 1: $\forall m, n \in \mathbb{Z}^{>1}$: $\|m\| \leqslant \left(\frac{\log m}{\log n} + 1\right) \cdot n \cdot \max\left\{\|n\|^{\frac{\log m}{\log n}}, 1\right\}$.

Claim 2: $\forall m, n \in \mathbb{Z}^{>1}$: $\|m\| \leqslant \max\left\{\|n\|^{\frac{\log m}{\log n}}, 1\right\}$

$\|n\| \geq 1$?

Claim 3: $\forall m, n \in \mathbb{Z}^{>1}$: $\|m\|^{\frac{1}{\log m}} = \|n\|^{\frac{1}{\log n}}$.

Write $m = \sum_{i=0}^{k} a_i n^i$ with $a_i \in \{0, \dots, n-1\}$, $a_k \neq 0$.

$$\Rightarrow \|m\| \leq \sum_{i=0}^{k} \underbrace{\|a_i\|}_{\leq n} \cdot \|n\|^i \leq n \cdot (k+1) \cdot \max\{1, \|n\|^k\} \leq n \cdot \left(\frac{\log m}{\log n} + 1\right) \cdot \max\left\{1, \|n\|^{\frac{\log m}{\log n}}\right\}$$

$$n^k \leq m < n^{k+1}$$

$$\Rightarrow k \cdot \log n \leq \log m \leq (k+1) \log n$$

$$k \leq \frac{\log m}{\log n}$$

Claim 1 $\Rightarrow \forall \ell \geq 1$: $\|m\|^\ell = \|m^\ell\| \leq \left(\ell \cdot \frac{\log m}{\log n} + 1\right) \cdot n \cdot \max\left\{\|n\|^{\frac{\log m}{\log n}}, 1\right\}^\ell$

$\log m^\ell = \ell \cdot \log m$

$$\Rightarrow \|m\| \leq \sqrt[\ell]{\left(\ell \cdot \frac{\log m}{\log n} + 1\right) \cdot n} \cdot \max\left\{\|n\|^{\frac{\log m}{\log n}}, 1\right\}.$$

$$\ell \to \infty \quad \Rightarrow \quad \|m\| \le \max\left\{ \|n\|^{\frac{\log m}{\log n}}, 1 \right\}.$$

$\|\cdot\|$ archimedean $\Rightarrow \exists\, m : \|m\| > 1 \Rightarrow \|n\| > 1.$

$$\Rightarrow \quad \|m\| \le \|n\|^{\frac{\log m}{\log n}}$$

$$\Rightarrow \quad \|m\|^{\frac{1}{\log m}} \le \|n\|^{\frac{1}{\log n}} \qquad \text{(Claim 3)}$$

Symmetry $\Rightarrow \|m\|^{\frac{1}{\log m}} = c > 1$

$\Rightarrow \quad \|m\| = c^{\log m} = m^{\log c} \quad \Rightarrow \quad \|\cdot\|$ equivalent to $|\cdot|_\infty$.

$\|\cdot\|$ non-archimedean $\Rightarrow$ ultrametric $\Rightarrow \forall\, n \in \mathbb{Z} : \|n\| \le 1.$

$\|\cdot\|$ nontrivial $\Rightarrow \exists\, a \in \mathbb{Q}^\times : \|a\| \neq 1$

$\Rightarrow \exists\, p$ prime $: \|p\| \neq 1.$

$\Rightarrow |p| < 1.$

$p \in \mathfrak{p} := \{ n \in \mathbb{Z} : \|n\| < 1 \}$ is an ideal of $\mathbb{Z}$.

$1 \notin \mathfrak{p}$ because $\|1\| = 1. \quad \Rightarrow \quad \mathfrak{p} = (p).$

(p) max. ideal

$\forall\, n \in \mathbb{Z} \setminus p\mathbb{Z} : \|n\| = 1.$

$\forall\, n \in \mathbb{Z} : \|n\| = \|p\|^{\operatorname{ord}_p(n)}$

$\Rightarrow \|\cdot\|$ equivalent to $|\cdot|_p$.

qed.