8.5 Completion of a metric space

$\mathbb{R}$ complete

$]-1,1[$ not complete

Consider a metric space (X, d).

Definition 8.5.1: A sequence (x_n) in X is ...

(a) ... said to *converge to* $x \in X$ and we write $x = \lim_{n\to\infty} x_n$, if

$$\forall\, \varepsilon > 0\ \exists\, n_0\ \forall\, n > n_0 \colon d(x_n, x) < \varepsilon.$$

(b) ... called a *Cauchy sequence* if

$$\forall\, \varepsilon > 0\ \exists\, n_0\ \forall\, n, m > n_0 \colon d(x_n, x_m) < \varepsilon.$$

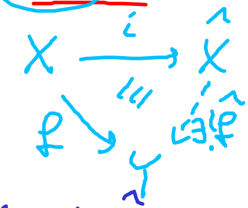
Proposition 8.5.2: Any convergent sequence is a Cauchy sequence and has a unique limit.

Definition 8.5.3: The metric space (X, d) is called *complete* if every Cauchy sequence has a limit.

Definition 8.5.4: A *completion of* (X, d) is a complete metric space $(\hat{X}, \hat{d})$ together with a map $i\colon X \to \hat{X}$ such that

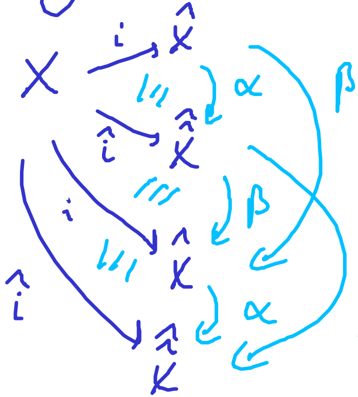
(a) for all $x, y \in X$ we have $\boxed{\hat{d}(i(x), i(y)) = d(x, y)}$ and $\Longrightarrow$ i injective.

(b) for every continuous map $f\colon X \to Y$ to a complete metric space (Y, e) there exists a unique continuous map $\hat{f}\colon \hat{X} \to Y$ such that $\hat{f} \circ i = f$.



Proposition 8.5.5: A completion exists and is unique up to unique isometry.

Proof: ① Uniqueness: $(\hat{\hat{X}}, \hat{\hat{e}})$ also a completion.



$\beta \circ \alpha = \mathrm{id}_{\hat{X}}$

$\alpha \circ \beta = \mathrm{id}_{\hat{\hat{X}}}$

$\Rightarrow \alpha, \beta$ inverse homeo's.

② Claim: $i(X)$ dense in $\hat{X}$.

Hence α, β are isometries.

Construct one completion with this property.

③ Define two sequences $(x_n), (y_n)$ in X to be equivalent if

$d(x_n, y_n) \to 0$ for $n \to \infty$.

Equivalence relation!

$(x_n) \sim (y_n) \Rightarrow \begin{cases} (x_n) \text{ Cauchy} \Rightarrow (y_n) \text{ Cauchy.} \\ x_n \to x \Rightarrow y_n \to x. \end{cases}$

④ Let $\hat{X} := \{\text{Cauchy sequences}\}/\sim$.

$\hat{d}([(x_n)], [(y_n)]) := \lim_{n\to\infty} d(x_n, y_n) \in \mathbb{R}^{\geq 0}$

$\Rightarrow$ metric space

5 $(\hat{X}, \hat{d})$ is complete: $\quad [(x_n^{(i)})_n]$ Cauchy sequence in $\hat{X}$

$$\Rightarrow \quad (x_n^{(n)})_n \quad \text{" " "} \quad X$$

and $$\lim_{i\to\infty} [(x_n^{(i)})_n] = [(x_n^{(n)})].$$

6 (a), (b)

7 Finish 2 $i(X)$ dense in $\hat{X}$:

$$\underset{i(x_n)}{\underset{\parallel}{\lim_{n\to\infty} [(\text{constant sequence } x_n)]}} = [(x_n)_n].$$

qed.

Reminder:

Definition 8.4.1: A *(non-trivial) absolute value* (or *norm*) on a field K is a map

$$K \longrightarrow \mathbb{R}^{\geqslant 0}, \quad x \mapsto |x|$$

with the properties

(a) For any $x \in K$ we have $|x| = 0$ if and only if $x = 0$.

(b) For any $x, y \in K$ we have $|xy| = |x| \cdot |y|$.

(c) For any $x, y \in K$ we have $|x + y| \leqslant |x| + |y|$.

(d) There exists $x \in K$ with $|x| \notin \{0, 1\}$.

8.6 Complete absolute values

Definition 8.6.1: An absolute value on a field is *complete* if and only if the associated metric space is complete.

Proposition 8.6.2: The completion of a field K with an absolute value $|\ |$ is a complete field $\hat{K}$ with the operations

$$\left(\lim_{n\to\infty} x_n\right) + \left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} (x_n + y_n)$$

$$\left(\lim_{n\to\infty} x_n\right) \cdot \left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} (x_n \cdot y_n)$$

and the absolute value

$$\left|\lim_{n\to\infty} x_n\right| = \lim_{n\to\infty} |x_n|$$

Example 8.6.3: The field $\mathbb{R}$ is the completion of $\mathbb{Q}$ for the absolute value $|\ |_\infty$.

i injective $\Rightarrow 0 \neq 1$ in $\hat{K}$.

$[(x_n)_n] \neq 0$

$\Rightarrow |[(x_n)_n]| > 0$.

$\Rightarrow \forall n \gg 0: |x_n| \geq \varepsilon$

$\frac{1}{|x_n|} \leq \frac{1}{\varepsilon}$.

$\Rightarrow \left(\frac{1}{x_n}\right)$ is a Cauchy seq.

Theorem 8.6.4: *(Ostrowski)* Any field that is complete with respect to an archimedean absolute value is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ and the absolute value is equivalent to the usual absolute value.

Proof: K archimedean $\Rightarrow \forall \xi \in \mathbb{R}\ \exists n \in \mathbb{Z} : |n \cdot 1| > \xi$. $\Rightarrow \operatorname{char}(K) = 0$.

$$\begin{tikzcd} K \arrow[dr, dash] & & \mathbb{C} \arrow[dl, dash] \\ & \mathbb{R} & \end{tikzcd}$$

K complete $\Rightarrow$ $\left.\begin{array}{l} L := \text{closure of } \mathbb{Q} \\ \text{abs. value } |_{\mathbb{Q}} \text{ is equivalent to } |\cdot|_\infty \end{array}\right\} \Rightarrow L \cong \mathbb{R}$

To show: $[K/\mathbb{R}] \leq 2$.

Enough: $\forall \xi \in K$: ξ is a zero of a quadratic polynomial over $\mathbb{R}$. Fix ξ.

For any $z \in \mathbb{C}$ set $\boxed{f(z) := |\xi^2 - \underbrace{(z+\bar{z})}_{\in \mathbb{R}}\xi + \underbrace{z\bar{z}}_{\in \mathbb{R}}|} \in \mathbb{R}^{\geq 0}$ $\in K$

continuous $\mathbb{C} \to \mathbb{R}^{\geq 0}$.

If $|z| \gg 0$ then $f(z) \geq |z|^2 + O(|z|)$ $\Rightarrow$ $f(z) \to \infty$ with $|z| \to \infty$.

$\Rightarrow f$ attains its minimum $m \geq 0$. If $m = 0$ we are done.

Assume $m > 0$. Then $S := \{z \in \mathbb{C} : f(z) = m\}$ = closed and bounded.

Choose $z_0 \in S$ with $|z_0|$ maximal. Choose $0 < \varepsilon < m$ and set

$$g(X) := \underbrace{X^2 - \underbrace{(z_0+\bar{z}_0)}_{\in \mathbb{R}}X + \underbrace{z_0\bar{z}_0}_{\in \mathbb{R}}} + \varepsilon \in \mathbb{R}[X].$$

Let z_1, z_1' be its roots in $\mathbb{C}$ with $|z_1| \geq |z_1'|$. Then $z_1 \cdot z_1' = z_0 \bar{z}_0 + \varepsilon$

$\Rightarrow |z_1| > |z_0|$

Choice of $z_0 \Rightarrow z_1 \notin S \Rightarrow \underline{\rho(z_1) > m}$.

Take $n \geq 1$, set $\underline{G(X) := (g(X) - \varepsilon)^n - (-\varepsilon)^n} = \prod_{i=1}^{2n} (X - \alpha_i)$, $\alpha_i \in \mathbb{C}$.

$g \mid G \Rightarrow$ WLOG: $z_1 = \alpha_1$

$$G(X)^2 = \prod_{i=1}^{2n} \left(X^2 - (\alpha_i + \bar{\alpha}_i) X + \alpha_i \cdot \bar{\alpha}_i \right) \in \mathbb{R}[X]$$

$$|G(\xi)|^2 = \prod_{i=1}^{2n} \left| \xi^2 - (\alpha_i + \bar{\alpha}_i) \xi + \alpha_i \bar{\alpha}_i \right| = \prod_{i=1}^{2n} \underbrace{\rho(\alpha_i)}_{\geq m} \geq \underset{\rho(z_1)}{\overset{\shortparallel}{\rho(\alpha_1)}} \cdot m^{2n-1}$$

$\rho(\alpha_1) = \rho(z_1) > m$

$$|G(\xi)| \leq \left| (g(\xi) - \varepsilon)^n - (-\varepsilon)^n \right| \leq \left| \underline{g(\xi) - \varepsilon} \right|^n + \varepsilon^n$$

$$= \rho(z_0)^n + \varepsilon^n = m^n + \varepsilon^n$$

$\Rightarrow (m^n + \varepsilon^n)^2 \geq \rho(z_1) \cdot m^{2n-1}$

$\Rightarrow \dfrac{\rho(z_1)}{m} \leq \left(1 + \left(\dfrac{\varepsilon}{m} \right)^n \right)^2$

Let $n \to \infty \Rightarrow \dfrac{\rho(z_1)}{m} \leq 1 \Rightarrow$ ↯.

qed.

Reminder:

Definition 8.4.8: An absolute value $|\ |$ is called *ultrametric* if it satisfies the stronger property

(c′) For any $x, y \in K$ we have $|x+y| \leqslant \max\{|x|, |y|\}$.

Proposition 8.4.9: (a) For any valuation v on K and any constant $0 < c < 1$ the map $|x| := c^{v(x)}$ is an ultrametric absolute value on K.

(b) Any ultrametric absolute value arises in this fashion from a valuation.

Proposition 8.6.5: An ultrametric absolute value $|\ |$ on a field K is complete if and only if the associated valuation v is complete.

$\mathcal{O}_K := \{x \in K : v(x) \geq 0\} = \{x \in K : |x| \leq 1\}$. $\Rightarrow$ $\forall x \in K \setminus \mathcal{O}_K : \frac{1}{x} \in \mathcal{O}_K$.

For any $0 < \varepsilon \leq 1$: $B_\varepsilon := \{x \in K : |x| \leq \varepsilon\}$ is an ideal of $\mathcal{O}_K$.

v complete $\Longleftrightarrow$ $\mathcal{O}_K \xrightarrow{\sim} \underbrace{\varprojlim_{\varepsilon} \mathcal{O}_K/B_\varepsilon}_{=:\hat{\mathcal{O}}_K} := \Big\{ (x_\varepsilon + B_\varepsilon)_\varepsilon \in \bigtimes_\varepsilon \mathcal{O}_K/B_\varepsilon \;\Big|\; \forall\, 0 < \varepsilon < \delta \leq 1 : x_\varepsilon \equiv x_\delta \bmod B_\delta \Big\}$.

$|(x_\varepsilon + B_\varepsilon)_\varepsilon| := \lim_{\varepsilon \to 0} |x_\varepsilon|$.

If all $x_\varepsilon \in B_\varepsilon$, then $|x_\varepsilon| \leq \varepsilon \Rightarrow \lim = 0$.

Otherwise $\exists \delta$: $x_\delta \notin B_\delta$. Then $\forall \varepsilon < \delta$: $x_\varepsilon \in x_\delta + B_\delta \Rightarrow |x_\varepsilon| = |x_\delta| \Rightarrow \lim > 0$.

$\ldots \Rightarrow$ ultrametric norm on $\hat{\mathcal{O}}_K$

Extend to $\hat{K} := \bigcup_{x \in K} \underbrace{x \cdot \hat{\mathcal{O}}_K}_{\text{within } \mathrm{Quot}(\hat{\mathcal{O}}_K)}$ $\Big\}$ Then $\hat{K}$ complete w.r.t. $|\cdot|$.

$\hat{\mathcal{O}}_K$ complete.

Take a Cauchy sequ. $(x_\varepsilon^{(n)} + B_\varepsilon)_\varepsilon$ in $\hat{\mathcal{O}}_K$.

Then $\forall \varepsilon$: $(x_\varepsilon^{(n)} + B_\varepsilon)_n$ eventually becomes constant, say $y_\varepsilon + B_\varepsilon$.

$\Rightarrow$ converges to $(y_\varepsilon + B_\varepsilon)_\varepsilon \in \hat{\mathcal{O}}_K$.

Conversely: If $|\cdot|$ is complete, take $(x_\varepsilon + B_\varepsilon)_\varepsilon \in \hat{\mathcal{O}}_K$

then $(x_{\frac{1}{n}})$ is a Cauchy sequ. in K $\Rightarrow$ converges to x.

$\Rightarrow$ $x + B_\varepsilon = x_\varepsilon + B_\varepsilon$ for all $\varepsilon > 0$. $\Rightarrow$ $\mathcal{O}_K \cong \hat{\mathcal{O}}_K$.

qed