8.7 Power series

ultrametric

Fix a complete nonarchimedean absolute value $|\ |$ associated to the valuation v on K.

Proposition 8.7.1: (a) A series $\sum_{n\geqslant 0} x_n$ in K converges if and only if $\lim_{n\to\infty} x_n = 0$.

(b) Convergent series in K can be arbitrarily rearranged and subdivided without changing convergence or the limit.

Ex.

Proof (a): $S_m := \sum_{n=0}^{m} x_n \Rightarrow \lim_{m\to\infty}(S_m - S_{m-1}) = \lim_{m\to\infty} S_m - \lim_{m\to\infty} S_{m-1} = 0$

$\lim_{m\to\infty}(S_m - S_{m-1}) = x_m$, and the second "=" holds if conv.

Suppose $\lim_{n\to\infty} x_n \to 0$. To prove: (S_m) is Cauchy.

$\forall \varepsilon > 0 \ \exists n_0 : \forall n \geq n_0 : |x_n| \leq \varepsilon$.

$\Rightarrow \forall m \geq n \geq n_0 : |S_m - S_n| = \left|\sum_{k=n+1}^{m} x_k\right| \overset{!}{\leq} \max\{\underbrace{|x_k|}_{\leq \varepsilon} : n < k \leq m\} \leq \varepsilon.$

qed.

uniformizer

Proposition 8.7.2: If v is normalized discrete, fix an element $\pi \in K$ with $v(\pi) = 1$ and a set of representatives $\mathcal{R}$ of $\mathcal{O}_v/\mathfrak{m}_v$ with $0 \in \mathcal{R}$. Then:

(a) Every element $x \in K$ can be written uniquely as a convergent Laurent series

$$\boxed{x = \sum_{i\in\mathbb{Z}} a_i \pi^i}$$

with $a_i \in \mathcal{R}$ and $a_i = 0$ for all $i \ll 0$.

(b) Such an element lies in $\mathcal{O}_v$ if and only if $a_i = 0$ for all $i < 0$.

$\mathcal{O}_v = \{ x \in K : v(x) \geq 0 \}$

$\mathfrak{m}_v = \{ \dots : v(x) > 0 \}$

$= \mathcal{O}_v \cdot \pi$.

Pf: $\mathcal{O}_v = \varprojlim_n \mathcal{O}_v/\mathfrak{m}_v^n$

$x \mapsto x + \mathfrak{m}_v^n = \sum_{i=0}^{n-1} a_i \pi^i + \mathfrak{m}_v^n$ with $a_i \in \mathcal{R}$ unique.

$x + \mathfrak{m}_v^{n+1} = \sum_{i=0}^{n} a_i' \pi^i + \mathfrak{m}_v^{n+1}$ ... $a_i' \in \mathcal{R}$ unique.

$\Rightarrow a_i = a_i'$ for $i < n$.

$x = \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} a_i \pi^i \right) = \sum_{i\geq 0} a_i \pi^i$.

$\underbrace{\mathfrak{m}_v^n}_{} = \pi^n \cdot \underbrace{\mathcal{O}_v}_{\text{complete}}$.

Conversely $\sum_{i\geq 0} a_i \pi^i + \mathfrak{m}_v^n = \sum_{i=0}^{n-1} a_i \pi^i + \mathfrak{m}_v^n$. $\Rightarrow a_i$ unique.

$K = \bigcup_{m\geq 0} \pi^{-m} \cdot \mathcal{O}_v \Rightarrow$ ditto.

qed.

Now we assume that $\mathbb{Q} \subset K$ and that the restriction of $|\ |$ to $\mathbb{Q}$ is $|\ |_p$. $\Longrightarrow$ $\mathbb{Q}_p \subset K$.

Proposition 8.7.3: For any $x \in K$ with $|x-1| < 1$ the series

$$|n|_p = p^{-\mathrm{ord}_p(n)} \geq \frac{1}{n}$$

$$\boxed{\log(x) \;:=\; \sum_{n\geqslant 1} (-1)^{n-1} \cdot \frac{(x-1)^n}{n}}$$

converges and satisfies

$$\log(xy) \;=\; \log(x) + \log(y).$$

Proof: $\left| \pm \frac{(x-1)^n}{n} \right| = \frac{|x-1|^n}{|n|} \leq n \cdot |x-1|^n \longrightarrow 0$ for $n \to \infty$ $\Longrightarrow$ Series converges.

$$\log(xy) = \sum_{n\geq 1} (-1)^{n-1} \cdot \frac{(xy-1)^n}{n} = \sum_{n\geq 1} (-1)^{n-1} \cdot \frac{[(x-1)(y-1)+(x-1)+(y-1)]^n}{n}$$

$$= \sum_{\substack{i,j,k\geq 0 \\ i+j+k\geq 1}} \frac{(-1)^{i+j+k-1}}{i+j+k} \cdot \binom{i+j+k}{i,j,k} \cdot (x-1)^{i+j} (y-1)^{i+k}$$

Lemma 8.7.4: For every $n \geqslant 0$ we have $\mathrm{ord}_p(n!) = \sum_{i\geqslant 1} \left\lfloor \frac{n}{p^i} \right\rfloor < \frac{n}{p-1}$.

$$\log(x) + \log(y) = \sum_{m,n\geq 0} \overset{\mathbb{Q}}{b_{m,n}} \cdot (x-1)^m \cdot (y-1)^n. \qquad = \sum_{m,n\geq 0} \underset{\mathbb{Q}}{a_{mn}} \cdot (x-1)^m \cdot (y-1)^n.$$

Enough to prove: $\forall m,n: b_{m,n} = a_{m,n}$

This holds because the identity holds over $\mathbb{R}$ with positive radius of convergence. qed.

Proof: Induction on n. $\operatorname{ord}_p(0!) = \operatorname{ord}_p(1) = 0$ ✓

$$\operatorname{ord}_p(n!) = \operatorname{ord}_p(n) + \operatorname{ord}_p((n-1)!)$$

$n = p^k m$, $p \nmid m$

$$= k + \sum_{i \geq 1} \left\lfloor \frac{n-1}{p^i} \right\rfloor$$

$$\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n-1}{p^i} \right\rfloor = \left\lfloor \frac{p^k m}{p^i} \right\rfloor - \left\lfloor \frac{p^k m - 1}{p^i} \right\rfloor = \begin{cases} i \leq k \Rightarrow 1 \\ i > k \Rightarrow 0 \end{cases}$$

$$= \sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor$$

qed.

Proposition 8.7.5: For every $x \in K$ with $|x| < p^{-\frac{1}{p-1}}$ the series

$$\boxed{\exp(x) := \sum_{n \geqslant 0} \frac{x^n}{n!}} = 1 + x + \sum_{n \geq 2} \frac{x^n}{n!}$$

converges and satisfies

$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

as above.

Proof: $\left| \frac{x^n}{n!} \right| < \frac{|x|^n}{p^{-\frac{n}{p-1}}} = \left(\frac{|x|}{p^{-\frac{1}{p-1}}} \right)^n$

$|n!| > p^{-\frac{n}{p-1}}$

qed

Also: $\exp \circ \log = \mathrm{id}$ and $\log \circ \exp = \mathrm{id}$.

Proposition 8.7.6: Exp and log induce mutually inverse group isomorphisms

$$(K,+) > \left\{x \in K : |x| < p^{-\frac{1}{p-1}}\right\} \underset{\exp}{\overset{\log}{\cong}} \left\{x \in K^\times : |x-1| < p^{-\frac{1}{p-1}}\right\} < (K^\times, \cdot).$$

Also: $(1+x)^a = \sum_{n \geq 0} \binom{a}{n} \cdot x^n$ for all $x, a \in K$ with $|x| < p^{\frac{-1}{p-1}}$, $|a| \leq 1$.

$$\binom{a}{n} = \frac{a(a-1) \cdots (a-n+1)}{n!}$$

Example 8.7.7: Exp and log induce mutually inverse group isomorphisms

$$\begin{aligned}(p\mathbb{Z}_p,+) &\cong (1+p\mathbb{Z}_p,\cdot) \quad \text{if } p>2,\\ (4\mathbb{Z}_p,+) &\cong (1+4\mathbb{Z}_2,\cdot) \quad \text{if } p=2.\end{aligned}$$

$K = \mathbb{Q}_p$.

$|x| < p^{\frac{-1}{p-1}} = \begin{cases} p^{-1} & \text{if } p=2 \\ > p^{-1} & \text{if } p>2 \end{cases} \iff \begin{aligned} \mathrm{ord}_p(x) &> 1 && \text{if } p=2 \\ &\geq 1 && \text{if } p>2. \end{aligned}$