9 Extensions of valuations

Throughout we fix a field K with an absolute value $|\ |$.

9.1 Normed vector spaces

Definition 9.1.1: A *norm* on a K-vector space V is a map $\|\ \|: V \to \mathbb{R}^{\geqslant 0}$ such that

(a) For any $v \in V$ we have $\|v\| = 0$ if and only if $v = 0$.

(b) For any $v \in V$ and $x \in K$ we have $\|xv\| = |x| \cdot \|v\|$.

(c) For any $v, w \in V$ we have $\|v + w\| \leqslant \|x\| + \|y\|$.

Definition 9.1.2: Two norms $\|\ \|$ and $\|\ \|'$ on V are called *equivalent* if there exist constants $c, c' > 0$ such that

$$\forall\, v \in V: \ \ c \cdot \|v\| \leqslant \|v\|' \leqslant c' \cdot \|v\|.$$

This is an equivalence relation on all norms on V.

Theorem 9.1.3: If K is complete and $\dim_K(V) < \infty$, any norms on V are equivalent. In particular V is then complete with respect to the metric induced by $\|\ \|$.

Proof: Take basis $v_1, \dots, v_n$ of V and $|\sum_{i=1}^{n} x_i v_i| := \max\{|x_i| : i = 1..n\}$

To show: $\|\cdot\|$ equivalent to $|\cdot|$. If so, then it follows that $\|\cdot\|$ is complete, because $|\cdot|$ is complete.

$$\Big\|\sum_i x_i v_i\Big\| \le \sum_i \|x_i v_i\| = \sum_i |x_i|\cdot\|v_i\| \le \underbrace{\Big(\sum_i \|v_i\|\Big)}_{\text{Constant.}}\cdot \Big|\sum x_i v_i\Big|.$$

Other direction: induction on n.

$n = 0$ ✓

$n > 0$ ⇝ let $V_i := \sum_{j \neq i} K\cdot v_j$. This has dimension $n-1$.

Induction hypothesis $\Rightarrow$ $\|\cdot\|\big|_{V_i}$ equivalent to $|\cdot|\big|_{V_i}$ $\Rightarrow$ $\|\cdot\|\big|_{V_i}$ complete $\Rightarrow$ V_i is closed in V w.r.t. $\|\cdot\|$.

$\Rightarrow$ $v_i \in V \setminus V_i$ = open w.r.t. $\|\cdot\|$. $\Rightarrow$ $\exists\, \varepsilon > 0$: $B_\varepsilon(v_i) \subset V \setminus V_i$.

Can take ε independent of i. Then $\forall w \in V$: $\|w\| < \varepsilon \Rightarrow \forall i$: $v_i + w \in V \setminus V_i$.

I.E.: $\boxed{\forall\, w_i \in V_i : \|v_i + w_i\| \geq \varepsilon}$

$(v_i + B_\varepsilon) \cap V_i = \emptyset$

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Now take $v = \sum_i x_i v_i \in V \setminus \{0\}$, take i with $|x_i|$ maximal. $\Rightarrow$ $|v| = |x_i|$.

$$\Rightarrow\quad x_i^{-1} v = \sum_j \frac{x_j}{x_i} v_j = v_i + \underbrace{\sum_{j \neq i} \frac{x_j}{x_i}\cdot v_j}_{\in V_i}.$$

$$\Rightarrow\quad \underbrace{\|x_i^{-1} v\|}_{\overset{!}{=}\ |x_i|^{-1}\cdot\|v\|} \geq \varepsilon. \quad\Rightarrow\quad \|v\| \geq \varepsilon\cdot|x_i| = \varepsilon\cdot|v|.$$

Counterexample: $K = \mathbb{Q}$, $|\cdot|_\infty$. $\|(x,y)\| := |x + \sqrt{2}\cdot y|$ on $\mathbb{Q}^2$ not equivalent to $|(x,y)| := \max\{|x|,|y|\}$.

qed.

9.2 Extensions of complete absolute values

Assume that $|\ |$ is complete.

Proposition 9.2.1: If $|\ |$ is archimedean, it possesses a unique extension to any algebraic extension of K. It is given by the formula $|y| \;=\; |\operatorname{Nm}_{L/K}(y)|^{1/[L/K]}$, and is again archimedean and complete.

Proof: $K = \mathbb{R}$ or $\mathbb{C}$, $|\cdot| = |\cdot|_\infty^{s}$ for some $0 < s \leq 1$.

$\Rightarrow$ $L = K$ and the statement is clear. ✓

or

$L = \mathbb{C}$, $K = \mathbb{R}$ and $|\cdot|_\infty^{s}$ is an extension, unique.

$|z| = |z\bar{z}|^{\frac{1}{2}}$

qed.

For the rest of this section we assume that $|\ |$ is complete and nonarchimedean. Let $\mathcal{O}_\mathfrak{p}$ be its valuation ring with the maximal ideal $\mathfrak{p}$ and the residue field $k := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$.

Definition 9.2.2: For $f(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$ we set

$$|f| \;:=\; \max\{|a_i| : 0 \leqslant i \leqslant n\}.$$

We call f *primitive* if $|f| = 1$. $\iff f \in \mathcal{O}_\mathfrak{p}[X] \setminus \mathfrak{p}[X]$.

Proposition 9.2.3: *(Hensel's Lemma)* Consider a primitive $f \in \mathcal{O}_{\mathfrak{p}}[X]$ and a decomposition $(f \bmod \mathfrak{p}) = \bar{g} \cdot \bar{h}$ with coprime polynomials $\bar{g}, \bar{h} \in k[X]$. Then there exist $g, h \in \mathcal{O}_{\mathfrak{p}}[X]$ with $(g \bmod \mathfrak{p}) = \bar{g}$ and $(h \bmod \mathfrak{p}) = \bar{h}$ and $\deg(g) = \deg(\bar{g})$ and $f = g \cdot h$.

Proof: Set $d := \deg(f)$, $m := \deg(\bar{g}) \Rightarrow \deg(\bar{h}) = \deg(\bar{f}) - m \leq d - m$.

Choose $g_0, h_0 \in \mathcal{O}_{\mathfrak{p}}[X]$ with $(g_0 \bmod \mathfrak{p}) = \bar{g}$ and $(h_0 \bmod \mathfrak{p}) = \bar{h}$ and $\deg g_0 = m$.

Since $\bar{g}$ and $\bar{h}$ are coprime, there exist $\bar{a}, \bar{b} \in k[X]$ with $\bar{a}\bar{g} + \bar{b}\bar{h} = 1$.

Lift $\bar{a}, \bar{b}$ to $a, b \in \mathcal{O}_{\mathfrak{p}}[X]$. Then $f - g_0 h_0 \in \mathfrak{p}[X]$ and $a g_0 + b h_0 - 1 \in \mathfrak{p}[X]$.

Take some coeff. π of these $\Rightarrow \pi \in \mathfrak{p}$ and $f - g_0 h_0 \in \pi \mathcal{O}_{\mathfrak{p}}[X]$ and $a g_0 + b h_0 - 1 \in \pi \mathcal{O}_{\mathfrak{p}}[X]$.

Now $|\pi| < 1$. Will find $p_i, q_i \in \mathcal{O}_{\mathfrak{p}}[X]$ for $i \geq 1$ such that $\deg(p_i) \leq m$ and $\deg(q_i) \leq d - m$

$$g := g_0 + p_1\pi + p_2\pi^2 + \ldots + p_i\pi^i + \ldots \in \mathcal{O}_{\mathfrak{p}}[X] \quad \text{with } \deg(g) = m$$
$$h := h_0 + q_1\pi + q_2\pi^2 + \ldots \in \mathcal{O}_{\mathfrak{p}}[X] \quad \text{with } \deg(h) \leq d - m.$$

Set
$$g_{n-1} := g_0 + p_1\pi + \ldots + p_{n-1}\pi^{n-1}$$
$$h_{n-1} := h_0 + q_1\pi + \ldots + q_{n-1}\pi^{n-1}$$

partial sums.

Want: $\boxed{f \equiv g_{n-1} h_{n-1} \bmod (\pi^n)}$.

Then in the limit we have $f \equiv gh \bmod (\pi^n)$
$\Rightarrow f = gh$.

$n = 1$: ✓ ok by construction.

$1 \leq n \to n+1$: $\quad g_n = g_{n-1} + p_n \pi^n$
$\qquad h_n = h_{n-1} + q_n \pi^n$

$$f - g_n h_n = (f - g_{n-1} h_{n-1}) - (g_{n-1} q_n + h_{n-1} p_n)\pi^n - \underbrace{p_n q_n \pi^{2n}}_{\equiv 0 \bmod (\pi^{n+1})}$$

want: $\equiv 0 \bmod (\pi^{n+1})$ $\qquad f - g_{n-1}h_{n-1} \in \pi^n \mathcal{O}_g[X]$ by assumption.

$$\iff \underbrace{\frac{f - g_{n-1} h_{n-1}}{\pi^n}}_{= \boxed{k_n}} \equiv g_{n-1} q_n + h_{n-1} p_n \bmod (\pi)$$

$$\equiv \boxed{g_0 q_n + h_0 p_n} \bmod (\pi)$$

But $k_n \equiv (a g_0 + b h_0) k_n \bmod (\pi)$

$$= g_0 \cdot a k_n + h_0 \cdot b k_n$$

Polynomial division: $b k_n = r_n \cdot g_0 + p_n$ with $p_n, r_n \in \mathcal{O}_g[X]$ and $\deg(p_n) \leq m$.

$$= g_0 \cdot a k_n + h_0 (r_n g_0 + p_n)$$

$$= g_0 \underbrace{(a k_n + h_0 r_n)}_{q_n'} + \underbrace{h_0 p_n}$$

Also $\deg(k_n) \leq d$. $\quad \deg(h_0 p_n) \leq d$ $\Big\} \implies \deg(q_n' \bmod (\pi)) \leq d - m$.

Remove from q_n' all terms of degree $> d-m$ to get q_n. Then $\deg(q_n) \leq d - m$.

qed.

Corollary 9.2.4: Consider a primitive $f \in \mathcal{O}_\mathfrak{p}[X]$ whose reduction modulo $\mathfrak{p}$ possesses a simple root $\alpha \in k(\mathfrak{p})$. Then f possesses a unique root $a \in \mathcal{O}_\mathfrak{p}$ with $a + \mathfrak{p} = \alpha$.

Proof: Take $\bar{g} = X - \alpha$ and $(f \bmod \mathfrak{p}) = \bar{g}\bar{h}$

$\Rightarrow \exists\, f = gh$ with $g \in \mathcal{O}_\mathfrak{p}[X]$ of degree 1 and $(g \bmod \mathfrak{p}) = \bar{g}$.

$g = uX - v$ for $u \in \mathcal{O}_\mathfrak{p}^\times \Rightarrow a := \frac{v}{u} \in \mathcal{O}_\mathfrak{p}$ with $(a \bmod \mathfrak{p}) = \alpha$. and $f(a) = 0$.

uniqueness ....

qed.

Corollary 9.2.5: Consider any irreducible $f(X) = \sum_{i=0}^n a_i X^i \in K[X]$ with $a_n \neq 0$.

(a) Then $|f| = \max\{|a_0|, |a_n|\}$.

(b) If $|f| = 1$, then all irreducible factors of f mod $\mathfrak{p}$ are equivalent.

Replace f by $\frac{f}{a_i}$. Then $|f| = 1$.

Example: p, q distinct odd primes $\left(\frac{p}{q}\right) = 1$.

$\Rightarrow X^2 - p$ has 2 distinct roots in $\mathbb{F}_q$.

$\Rightarrow \sqrt{p} \in \mathbb{Z}_q$.

Proof (a) If not, say $|f| = |a_i|$ with i maximal, $0 < i < n$.

$\Rightarrow (f \bmod \mathfrak{p}) = \sum_{j=0}^{i} \bar{a}_j X^j =: \bar{g}$, $\bar{h} := 1$

Hensel $\Rightarrow$ $f = gh$ with $(g \bmod \mathfrak{p}) = \bar{g}$ and $\deg(g) = i$.

$\Rightarrow$ Contradiction irreducibility.

(b) similar.

qed

Theorem 9.2.6: Consider any finite field extension L/K of degree n.

(a) There exists a unique absolute value on L which extends $|\ |$.

(b) This is given by the formula $|y| = |\operatorname{Nm}_{L/K}(y)|^{1/n}$.

(c) This extension is again nonarchimedean and complete.

Corollary 9.2.7: For any algebraic extension L/K there exists a unique absolute value on L that extends $|\ |$. and it is nonarchimedean.

Proof: $L = \bigcup L'$, L'/K finite. Uniqueness $\Rightarrow$ unique & compatible extension to all L'. qed.

Caution: $[L/K] = \infty \Rightarrow$ extension not complete.

E.g.: $\overline{\mathbb{Q}}_p$ alg. closure