Reminder:

We fix a field K with a complete nonarchimedean absolute value $|\ |$ with valuation ring $\mathcal{O}_{\mathfrak{p}}$, with maximal ideal $\mathfrak{p}$, and with residue field $k := \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$.

Theorem 9.1.3: Any norms on a finite dimensional K-vector space are equivalent and complete.

Proposition 9.2.3: *(Hensel's Lemma)* Consider a primitive $f \in \mathcal{O}_{\mathfrak{p}}[X]$ and a decomposition $(f \bmod \mathfrak{p}) = \bar{g} \cdot \bar{h}$ with coprime polynomials $\bar{g}, \bar{h} \in k[X]$. Then there exist $g, h \in \mathcal{O}_{\mathfrak{p}}[X]$ with $(g \bmod \mathfrak{p}) = \bar{g}$ and $(h \bmod \mathfrak{p}) = \bar{h}$ and $\deg(g) = \deg(\bar{g})$ and $f = g \cdot h$.

Corollary 9.2.5: Consider any irreducible $f(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$ with $a_n \neq 0$.
Then $|f| = \max\{|a_0|, |a_n|\}$.

Theorem 9.2.6: Consider any finite field extension L/K of degree n.

(a) There exists a unique absolute value on L which extends $|\ |$.

(b) This is given by the formula $|y| = |\operatorname{Nm}_{L/K}(y)|^{1/n}$.

(c) This extension is again nonarchimedean and complete.

Any extension is a norm on the K-vector space L.
$\Rightarrow$ complete and unique up to equivalence.
$\Rightarrow$ unique up to a power.
$\Downarrow$ exponent $= 1$!
$\Rightarrow$ uniqueness.
qed.

Proof: Let $\tilde{\mathcal{O}}$ be the integral closure of $\mathcal{O}_{\mathfrak{p}}$ in L.

Claim: $\forall y \in L:\ y \in \tilde{\mathcal{O}} \iff \operatorname{Nm}_{L/K}(y) \in \mathcal{O}_{\mathfrak{p}}$.

Proof: "$\Rightarrow$": See earlier chapter.

"$\Leftarrow$": $f \in K[X]$ min. pol. of y $\Rightarrow$ f monic & irred. $f = \sum_{i=0}^{n} a_i X^i$ $\overset{9.2.5}{\Longrightarrow}$ $|f| = \max\{|a_0|, 1\}^{[L/K(y)]}$

$L/K(y)/K$, $\underbrace{\mathrm{Nm}_{L/K}(y)}_{\in \mathcal{O}_g} = \mathrm{Nm}_{K(y)/K}(\underbrace{\mathrm{Nm}_{L/K(y)}(y)}_{= y^{[L/K(y)]}}) = \underbrace{\mathrm{Nm}_{K(y)/K}(y)}_{= \pm a_0}$

$\Rightarrow a_0 \in \mathcal{O}_g \Rightarrow |a_0| \leq 1 \Rightarrow |f| = 1 \Rightarrow f \in \mathcal{O}_g[X]$.

$\Rightarrow y \in \tilde{\mathcal{O}}$.

qed.

For any $y \in L$ set $|y| := |\mathrm{Nm}_{L/K}(y)|^{1/[L/K]}$.

This is a map $L \to \mathbb{R}^{\geq 0}$ which extends $|\cdot|$.

$y \in K \Rightarrow \mathrm{Nm}_{L/K}(y) = y^{[L/K]}$.

(a) $|y| = 0 \iff \mathrm{Nm}_{L/K}(y) = 0 \iff y = 0$.

(b) $|xy| = |x| \cdot |y|$ because $\mathrm{Nm}_{L/K}(xy) = \mathrm{Nm}_{L/K}(x) \cdot \mathrm{Nm}_{L/K}(y)$.

(c) $|x+y| = \ldots$

WLOG $|x| \geq |y|$.

$\Rightarrow |\frac{y}{x}| \leq 1$

$\Rightarrow \mathrm{Nm}_{L/K}(\frac{y}{x}) \in \mathcal{O}_g \Rightarrow \frac{y}{x} \in \tilde{\mathcal{O}} \Rightarrow 1 + \frac{y}{x} \in \tilde{\mathcal{O}} \Rightarrow \mathrm{Nm}_{L/K}(1+\frac{y}{x}) \in \mathcal{O}_g$

$\Rightarrow |1 + \frac{y}{x}| \leq 1$.

$= |x| \cdot |1 + \frac{y}{x}| \leq |x| = \max\{|x|, |y|\}$.

9.3 Newton Polygons

$v(\alpha) = \frac{1}{n} \cdot v(a_0)$.

Assume that $|\ |$ is complete and nonarchimedean and associated to a valuation v.

Proposition 9.3.1: For any irreducible monic polynomial $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i \in K[X]$ and any zero α in an algebraic closure of K we have $|\alpha| = |a_0|^{1/n}$.

unique extension.

Proof: Let $L := K(\alpha) \Rightarrow f$ min. Pol. of α over $K \Rightarrow [L/K] = n$.

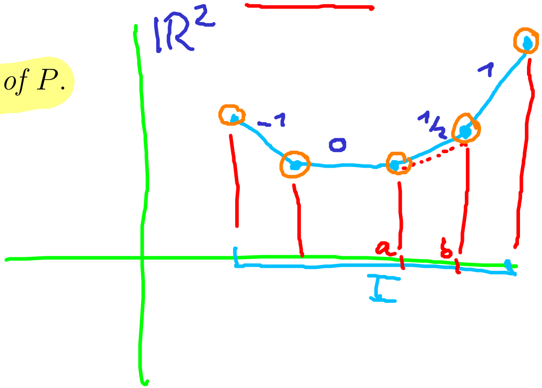
$$\Rightarrow |\alpha| = |N_{L/K}(\alpha)|^{\frac{1}{n}} = |\pm a_0|^{\frac{1}{n}} = |a_0|^{\frac{1}{n}}.$$

qed

Definition 9.3.2: (a) A *convex polygon $P \subset \mathbb{R}^2$* is the graph of a piecewise linear convex function $I \to \mathbb{R}$ for some closed interval I.

(b) If such P contains a straight line segment of slope ξ over the maximal interval $[a, b] \subset I$ with a, b, then ξ is called a *slope of P of multiplicity $b - a$.*

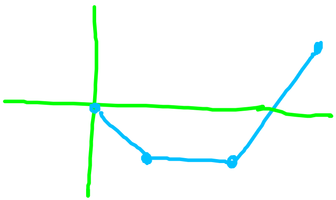
(c) A point on P where the slope changes is called a *break point of P*.

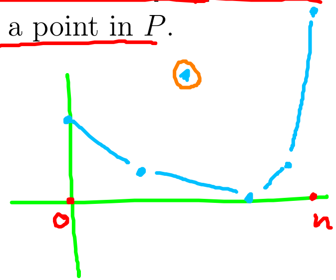


Fix a polynomial $f(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$ with $a_0, a_n \neq 0$ and consider the finite set

$$S := \{(i, v(a_i)) \mid 0 \leqslant i \leqslant n \text{ with } a_i \neq 0\}. \quad \subset \mathbb{R}^2$$

Definition 9.3.3: The *Newton polygon* of f is the unique convex polygon over the interval $[0, n]$ with all end points and break points in S, such that each point of S lies vertically above a point in P.





Proposition 9.3.4: Write $f(X) = a_n \cdot \prod_{i=1}^{n}(X - \alpha_i)$ with $\alpha_i \in \bar{K}$. Then for every real number ξ, the multiplicity of ξ as a slope of the Newton polygon of f is the number of i with $v(\alpha_i) = -\xi$.

Proof: Let $w_i := v(\alpha_i)$; WLOG: $w_1 \leq w_2 \leq \ldots \leq w_n$.

$$a_{n-i} = \pm a_n \cdot \sum_{1 \leq \nu_1 < \nu_2 < \ldots < \nu_i \leq n} \alpha_{\nu_1} \cdots \alpha_{\nu_i}$$

Suppose $i = 0$ or n or $w_i < w_{i+1}$. Then $v(\alpha_1 \cdots \alpha_i) = w_1 + \ldots + w_i < v(\alpha_{\nu_1} \cdots \alpha_{\nu_i})$ for all other terms

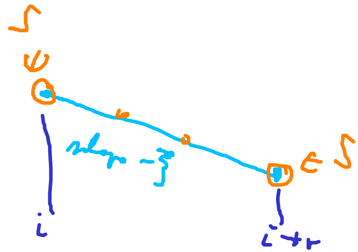
$$\Rightarrow v(a_{n-i}) = v(a_n) + w_1 + \ldots + w_i.$$

For arbitrary i we have $v(\alpha_{\nu_1} \cdots \alpha_{\nu_i}) \geq w_1 + \dots + w_i$

$\Rightarrow$ $v(a_{n-i}) \geq v(a_n) + w_1 + \dots + w_i$.

So if $w_i < w_{i+1} = \dots = w_{i+r} < w_{i+r+1}$ ($= \xi$)

r roots of value w_i.



See Th. 6.3. Neukirch.

qed.