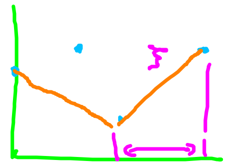
Reminder:

We fix a field K with a complete nonarchimedean absolute value $|\ |$ with valuation ring $\mathcal{O}_\mathfrak{p}$, with maximal ideal $\mathfrak{p}$, and with residue field $k := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$.

Take a polynomial $f(X) = \sum_{i=0}^n a_i X^i \in K[X]$ with $a_0, a_n \neq 0$ and consider the finite set

$$S := \{(i, v(a_i)) \mid 0 \leqslant i \leqslant n \text{ with } a_i \neq 0\}.$$



Definition 9.3.3: The *Newton polygon* of f is the unique convex polygon over the interval $[0, n]$ with all end points and break points in S, such that each point of S lies vertically above a point in P.

Proposition 9.3.4: Write $f(X) = a_n \cdot \prod_{i=1}^n (X - \alpha_i)$ with $\alpha_i \in \bar{K}$. Then for every real number ξ, the multiplicity of ξ as a slope of the Newton polygon of f is the number of i with $v(\alpha_i) = -\xi$.

Corollary 9.2.5: *(of Hensel's Lemma)* If f is irreducible, then $|f| = \max\{|a_0|, |a_n|\}$.

Proposition 9.3.5: The decomposition of the Newton polygon of f into straight line segments corresponds to a factorization $f = a_n \prod_\xi f_\xi$ over K with

$$\boxed{f_\xi(X) \;:=\; \prod_{\substack{1\leqslant i\leqslant n\\ v(\alpha_i)=-\xi}} (X-\alpha_i) \;\in\; K[X]}$$

for every real number ξ.

Proof: Enough for f irreducible. To show that all roots α_i have the same $v(\alpha_i)$.

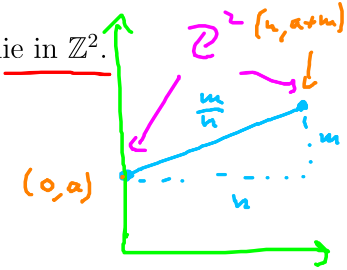
$L := K[X]/(f) \cong K(\alpha_i)$ for any i. $\qquad n = \deg(f) = [L/K]$.

$\Rightarrow v(\alpha_i) = \frac{1}{n} \cdot v\Big(\underbrace{Nm_{L/K}(\alpha_i)}_{= \pm \frac{a_0}{a_n}}\Big)$. = independent of i.

qed.

Now assume in addition that the valuation v is discrete & normalized. i.e. $v(K^\times) = \mathbb{Z}$.

Proposition 9.3.6: Then all end points and break points of the Newton polygon lie in $\mathbb{Z}^2$.



Pf.: Then $S \subset \mathbb{Z}^2$ qed.

Proposition 9.3.7: If the Newton polygon of f has a single slope of the form $\frac{m}{n}$ for an integer m coprime to $n = \deg(f)$, then f is irreducible.

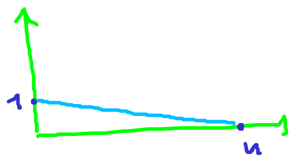
Pf: If g has slope $\frac{m}{n}$ with multiplicity $d > 0$ $\Rightarrow$ $d \cdot \frac{m}{n} \in \mathbb{Z}$ $\Rightarrow$ $n \mid d$.

$d \cdot \frac{m}{n} = \pm v\left(\frac{a_0}{a_n}\right)$. $\qquad n \mid d \Downarrow d \geq n$.

qed.

Note 9.3.8: In the case $m = -1$ this is the Eisenstein criterion.

f monic, degree n



9.4 Lifting prime ideals

As before we assume that $|\ |$ is complete and associated to a valuation v with valuation ring $\mathcal{O}_\mathfrak{p}$ and maximal ideal $\mathfrak{p}$ and residue field $k(\mathfrak{p}) := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$. Consider a separable finite extension L/K and let w denote the unique valuation on L that extends v.

Proposition 9.4.1: (a) The integral closure of $\mathcal{O}_\mathfrak{p}$ in L is the ring

$$\mathcal{O}_\mathfrak{q} := \{\overset{y}{\cancel{w}} \in L : \overset{w}{\cancel{v}}(y) \geqslant 0\}.$$

← See proof of Thm. 9.2.6.

int. clos $= \{y \in L \mid v(\mathrm{Nm}_{L/K}(y)) \geq 0\}$

$w(y) = \frac{1}{[L:K]} \cdot v(\mathrm{Nm}_{L/K}(y))$

(b) There is a unique prime $\mathfrak{q}$ of $\mathcal{O}_\mathfrak{q}$ above $\mathfrak{p}$, given by

$$\mathfrak{q} := \{y \in L : w(y) > 0\}.$$

$\|$

unique max. ideal of $\mathcal{O}_w$

$\square$

From Section 6.2 we obtain:

Proposition 9.4.2: Assume that $k(\mathfrak{p})$ is perfect. Then:

(a) The ramification degree of $\mathfrak{q}$ over $\mathfrak{p}$ is $e_{\mathfrak{q}|\mathfrak{p}} = [w(L^\times) : v(K^\times)]$.

(b) The inertia degree of $\mathfrak{q}$ over $\mathfrak{p}$ is $f_{\mathfrak{q}|\mathfrak{p}} = [k(\mathfrak{q})/k(\mathfrak{p})]$.

(c) We have $[L/K] = e_{\mathfrak{q}|\mathfrak{p}} \cdot f_{\mathfrak{q}|\mathfrak{p}}$.

With $\mathfrak{q} = \mathcal{O}_{\mathfrak{q}} \cdot (\Pi)$

$$\boxed{\mathfrak{p} \cdot \mathcal{O}_{\mathfrak{q}} = \mathfrak{q}^e}$$

for $e = e_{\mathfrak{q}|\mathfrak{p}}$.

$\Rightarrow w(L^\times) = \mathbb{Z} \cdot w(\Pi)$

$v(K^\times) = \mathbb{Z} \cdot v(\pi)$

$\mathfrak{p} = \mathcal{O}_{\mathfrak{p}} \cdot \pi$

$\pi \in \Pi^e \cdot \mathcal{O}_{\mathfrak{q}}^\times$

$\Rightarrow v(\pi) = e \cdot w(\Pi)$

qed.

9.5 Extensions of absolute values

Proposition 9.5.1: *(Simultaneous approximation)* Consider pairwise inequivalent norms $|\ |_1, \dots, |\ |_n$ on a field K and elements $a_1, \dots, a_n \in K$. Then for every $\varepsilon > 0$ there exists an $x \in K$ such that $|x - a_i|_i < \varepsilon$ for all i.

Proof: Recall $|\cdot|, |\cdot|'$ equivalent $\iff (\forall x \in K : |x| < 1 \iff |x|' < 1)$

$n = 0$ ✓

$n = 1$: Take $x = a_1$

$n \geq 2$: $|\cdot|_1, |\cdot|_2$ inequivalent $\Rightarrow \exists\, \alpha \in K : |\alpha|_1 < 1$ and $|\alpha|_n \geq 1$.
$\exists\, \beta \in K : |\beta|_1 \geq 1$ and $|\beta|_n < 1$

$\Rightarrow y := \frac{\beta}{\alpha}$ satisfies $|y|_1 > 1 > |y|_n$.

$\boxed{\text{why exactly?}}$

By the corrected form of 8.4.7 (b).

Claim: $\exists\, z \in K : |z|_1 > 1 > |z|_i$ for $2 \leq i \leq n$.

Proof: $n = 2$ ✓

$n > 2 \Rightarrow$ Take $z \in K$ with $|z|_1 > 1 > |z|_i$ for $2 \leq i \leq n-1$.

Case $|z|_n \leq 1$: Then $z^m y$ does it for $m \gg 0$:

$|z^m y|_1 = |z|_1^m \cdot |y|_1 > 1$

$|z^m y|_n = |z|_n^m \cdot |y|_n < 1$.

For $1 < i < n$: $|z^m y|_i = |z|_i^m \cdot |y|_i < 1$ for $m \gg 0$

Case $|z|_n > 1$: Let $t_m := \frac{z^m}{1 + z^m}$ for $m \gg 0$:

$|t_m|_i \to 0$ for $1 < i < n$

$|t_m|_i \to 1$ for $i = 1, n$.

Then $t_m y$ does it:

qed.

By the claim we can find $z_i \in K$ with $|z_i|_i > 1 > |z_i|_j$ for all $i \neq j$.

Then $$\frac{z_i^m}{1+z_i^m} \longrightarrow \begin{cases} 1 & \text{for } |\cdot|_i \\ 0 & \text{for } |\cdot|_j \text{ for } j \neq i \end{cases}.$$

Take $$x := \sum_i a_i \cdot \frac{z_i^m}{1+z_i^m} \quad \text{for } m \gg 0.$$

$$\Rightarrow \quad |x - a_i|_i \leq \left| a_i \cdot \Big(\underbrace{\frac{z_i^m}{1+z_i^m} - 1}_{\to 0} \Big) \right|_i + \sum_{j \neq i} \left| a_j \cdot \underbrace{\frac{z_j^m}{1+z_j^m}}_{\to 0} \right|_i < \varepsilon$$
for $m \gg 0$.

(qed)

Proposition 9.5.2: For any finite separable field extension L/K and any field extension $\hat{K}/K$ we have

$$\boxed{L \otimes_K \hat{K} \;\cong\; \bigtimes_{i=1}^{r} \hat{L}_i}$$

for finite separable field extensions $\hat{L}_i/\hat{K}$ with $[L/K] = \sum_{i=1}^{r} [\hat{L}_i/\hat{K}]$.

$$\forall i: \quad \begin{array}{ccc} L & \hookrightarrow & \hat{L}_i \\ | & & | \\ K & \hookrightarrow & \hat{K} \end{array}$$

Proof. Write $L = K(a) \cong K[X]/(f(X))$ for $f \in K[X]$ irreducible. separable

$$\Rightarrow L \otimes_K \hat{K} \cong \hat{K}[X]/(f(X)) \cong \bigtimes_{i=1}^{r} \underbrace{\hat{K}[X]/(f_i)}_{\cong\, \hat{L}_i \,=\, \text{finite sep. extn of } \hat{K}}$$

Chinese Remainder Thm.

$f = \prod_{i=1}^{r} f_i$ — irred. monic in $\hat{K}[X]$.

$\Rightarrow f_i$ pairwise inequivalent. separable

$[L/K] = \dim_K(L) = \dim_{\hat{K}}(L \otimes_K \hat{K}) = \sum \dim_{\hat{K}}(\hat{L}_i)$.

qed.

Now we fix an absolute value $|\ |$ on K that is not necessarily complete, and a finite separable extension L/K. We apply the above to the completion $\hat{K}$ of K with respect to $|\ |$ with the extended absolute value $|\ |\hat{}$. For each i let $|\ |\hat{}_i$ denote the unique absolute value on $\hat{L}_i$ that extends $|\ |$ on $\hat{K}$.

Lemma 9.5.3: Two absolute values on L that extend $|\ |$ are equivalent if and only if they are equal.

Proposition 9.5.4: The different extensions of $|\ |$ to L are precisely the restrictions $|\ |_i$ of the $|\ |\hat{}_i$ to L, and for each of them the completion of L is $\hat{L}_i$.

Proof: Each $|\cdot|\hat{}_i$ induces an abs. value on L that extends $|\cdot|$.

$$\begin{tikzcd} & \hat{L} \arrow[dl, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & & \hat{K} \arrow[dl, dash] \\ & K & \end{tikzcd}$$

Conversely: Let $|\cdot|'$ be an abs. value on L that extends $|\cdot|$. Let $\hat{L}$ be its completion of L with abs. value $|\cdot|\hat{}$

$\Rightarrow L \otimes_K \hat{K} \longrightarrow \hat{L}$ K-algebra homom.

$y \otimes z \longmapsto y \cdot z$

This must factor through $L \otimes_K \hat{K} \twoheadrightarrow \hat{L}_i \hookrightarrow \hat{L}$ for unique i

Now $|\cdot|\hat{}$ and $|\cdot|\hat{}_i$ both extend $|\cdot|$ on $\hat{K}$ $\Rightarrow$ $|\cdot|\hat{}$ restricts to $|\cdot|\hat{}_i$.

Also L has dense image in $\hat{L}$ $\Longrightarrow$ $\left.\begin{array}{l}\hat{L}_i \text{ dense in } \hat{L}. \\ \text{\& closed because complete}\end{array}\right\} \Rightarrow \underline{\hat{L}_i = \hat{L}.}$

<u>qed</u>

Proposition 9.5.5: Assume that $| \; |$ is archimedean. Then either

(a) $\hat{K} \cong \mathbb{C}$ and all $\hat{L}_i \cong \mathbb{C}$ and $[L/K] = r$.

(b) $\hat{K} \cong \mathbb{R}$ and all $\hat{L}_i \cong \mathbb{R}$ or $\mathbb{C}$ and $[L/K] = r_1 + 2r_2$, where r_1 is the number of i with $\hat{L}_i \cong \mathbb{R}$ and r_2 the number of i with $\hat{L}_i \cong \mathbb{C}$.

For the rest of this section we assume that $| \; |$ is nonarchimedean and corresponds to the discrete valuation $\mathrm{ord}_{\mathfrak{p}}$ for a maximal ideal $\mathfrak{p}$ of a Dedekind ring A with $\mathrm{Quot}(A) = K$. Let $\mathcal{O} \subset \hat{K}$ denote the respective completions and $\mathfrak{m} \subset \mathcal{O}$ the maximal ideal. By Proposition 9.4.1 the integral closure $\mathcal{O}_i \subset \hat{L}_i$ of $\mathcal{O}$ is the valuation ring for $| \; |_{\hat{i}}$. Let $\mathfrak{n}_i$ denote its unique maximal ideal. Let B denote the integral closure of A in L.

$L \otimes_K \hat{K} \cong \bigtimes_{i=1}^{r} \hat{L}_i$

Proposition 9.5.6: The isomorphism in Proposition 9.5.2 induces an isomorphism

$$\boxed{B \otimes_A \mathcal{O} \cong \bigtimes_{i=1}^{r} \mathcal{O}_i}$$

Proof: Replace A by its localization at $\mathfrak{p}$, B . . . . .] $\Rightarrow$ A valuation ring $\Rightarrow$ B free A-module of rank $n = [L/K]$.

$\Rightarrow$ $B \otimes_A \mathcal{O}$ is a free $\mathcal{O}$-module of rank n.

$B \otimes_A K = L$.

and $B \otimes_A \mathcal{O} \subset B \otimes_A \hat{K} = L \otimes_K \hat{K} = \bigtimes_i \hat{L}_i$

$\forall\, b \in B$: b integral over A $\Rightarrow$ $\forall i$: its image in $\hat{L}_i$ is integral over $\mathcal{O}$ $\Rightarrow$ lies in $\mathcal{O}_i$

$\Rightarrow$ $\boxed{B \otimes_A \mathcal{O} \xrightarrow{e} \bigtimes_{i=1}^{n} \mathcal{O}_i}$ $\mathcal{O}$-linear.

$n = \sum_{i=1}^{r} [\hat{L}_i / \hat{K}]$

Each $\mathcal{O}_i$ is a free $\mathcal{O}$-module of rank $[\hat{L}_i / \hat{K}]$.

Both sides are free $\mathcal{O}$-modules of rank n.

$\Rightarrow$ $\exists\, n \geq 0$: $e(B \otimes_A \mathcal{O}) \supset \bigtimes_{i=1}^{n} \mathfrak{m}^n \cdot \mathcal{O}_i$.

To show: This holds with $n = 0$.

Rest next time.