Reminder:

We fix a field K with a nonarchimedean valuation $\mathrm{ord}_{\mathfrak{p}}$ for a maximal ideal $\mathfrak{p}$ of a Dedekind ring A with $\mathrm{Quot}(A) = K$. Let $\mathcal{O} \subset \hat{K}$ denote the respective completions and $\mathfrak{m} \subset \mathcal{O}$ the maximal ideal. Let L/K be a separable finite extension $\cancel{L/K}$ and B the integral closure of A in L. For each extension of $\mathrm{ord}_{\mathfrak{p}}$ to a valuation on L let $\hat{L}_i$ be the associated completion with valuation ring $\mathcal{O}_i$ and maximal ideal $\mathfrak{n}_i$.

Propositions 9.5.2 and 9.5.6: We have a natural isomorphisms

$$\boxed{L \otimes_K \hat{K} \;\cong\; \bigtimes_{i=1}^{r} \hat{L}_i} \quad \text{and} \quad \boxed{B \otimes_A \mathcal{O} \;\cong\; \bigtimes_{i=1}^{r} \mathcal{O}_i.}$$

Proposition 9.5.7: (a) The prime ideals of B above $\mathfrak{p}$ are precisely the r different ideals $\mathfrak{q}_i := \mathfrak{n}_i \cap B$.

(b) For each of them we have $e_{\mathfrak{q}_i|\mathfrak{p}} = e_{\mathfrak{n}_i|\mathfrak{m}}$ and $f_{\mathfrak{q}_i|\mathfrak{p}} = f_{\mathfrak{n}_i|\mathfrak{m}}$.

Note 9.5.8: This explains the formula $[L/K] = \sum_{i=1}^{r} e_{\mathfrak{q}_i|\mathfrak{p}} \cdot f_{\mathfrak{q}_i|\mathfrak{p}}$ in terms of extensions of valuations.

Proposition 9.5.9: (a) The isomorphism in Prop. 9.5.6 induces an isomorphism

$$\boxed{\mathrm{diff}_{B/A} \otimes_A \mathcal{O} \;\cong\; \bigtimes_{i=1}^{r} \mathrm{diff}_{\mathcal{O}_i/\mathcal{O}}\,.}$$

(b) The embedding $A \subset \mathcal{O}$ induces an equality

$$\boxed{\mathrm{disc}_{B/A} \cdot \mathcal{O} \;=\; \prod_{i=1}^{r} \mathrm{disc}_{\mathcal{O}_i/\mathcal{O}}\,.}$$

$\mathrm{diff}^{-1}_{B/A} = \left\{ x \in L \;\middle|\; \begin{array}{l}\forall y \in B:\\ \mathrm{tr}(xy) \in A\end{array} \right\}$

$\mathrm{diff}^{-1}_{B/A} \times B \xrightarrow{\mathrm{tr}} A$

$\mathrm{diff}^{-1}_{B/A} \cong \mathrm{Hom}_A(B, A)$

$x \mapsto (y \mapsto \mathrm{tr}(xy))$.

$B \otimes_A \mathcal{O} \to \mathcal{O}$ $\quad \mathcal{O}$-lin

$\Longleftrightarrow$

$B \times \mathcal{O} \to \mathcal{O}$ $\quad$ A-bilin, $\mathcal{O}$-lin in $\mathcal{O}$

$\Longleftrightarrow$

$B \to \mathcal{O}$ $\quad$ A-lin

localize $A, B \Rightarrow$

WLOG: B free A-module.

$\mathrm{diff}_{B/A} = dB$ for $d \in L^\times$.

Proof: (a)

$$\begin{array}{ccccccccc}
L \otimes_K \hat{K} & \supset & \mathrm{diff}^{-1}_{B/A} \otimes_A \mathcal{O} & \xrightarrow{\sim} & \mathrm{Hom}_A(B, A) \otimes \mathcal{O} & & & & \\
 & & & & \| & & & & \\
 & & & & \mathrm{Hom}_A(B, \mathcal{O}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}}(B \otimes_A \mathcal{O}, \mathcal{O}) & & \\
\wr\| & /// & \wr\downarrow & & & & \wr\| & & \\
\bigtimes_{i=1}^{\hat{}} \hat{L}_i & \supset & \bigtimes_{i=1}^{r} \mathrm{diff}^{-1}_{\mathcal{O}_i/\mathcal{O}} & \xrightarrow{\sim} & \bigtimes_{i=1}^{\hat{}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_i, \mathcal{O}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}}\left(\bigtimes_{i=1}^{\hat{}} \mathcal{O}_i, \mathcal{O}\right) & &
\end{array}$$

$$\begin{array}{ccccc}
x \otimes y & \longmapsto & (z \mapsto \mathrm{tr}(\kappa z) \otimes y) & & \\
 & & \updownarrow & & \\
\downarrow & & z \longmapsto \mathrm{tr}(\kappa z) \cdot y & \longmapsto & \left(b \otimes c \mapsto \mathrm{tr}_{L/K}(xb) \cdot cy = \sum_i \mathrm{tr}_{L_i/\hat{K}}(xb) \cdot cy\right) \\
(\kappa y)_i & \longmapsto & & & \left((b_i)_i \mapsto \sum_i \mathrm{tr}_{L_i/\hat{K}}(\kappa y\, b_i)\right)
\end{array}$$

So the diagram commutes.

(b) $\mathrm{diff}_{B/A} = dB$

$\overset{(a)}{\Rightarrow} \mathrm{diff}_{\mathcal{O}_i/\mathcal{O}} = d\,\mathcal{O}_i$

$\mathrm{disc}_{B/A} = \mathrm{Nm}_{B/A}(\mathrm{diff}\, B/A) = \mathrm{Nm}_{L/K}(d)\cdot A.$

$\Rightarrow \mathrm{disc}_{B/A}\cdot\mathcal{O} = \mathrm{Nm}_{L/K}(d)\cdot\mathcal{O}$

$= \prod_i \mathrm{Nm}_{L_i/\hat{K}}(d)\cdot\mathcal{O}$

$= \prod_i \mathrm{disc}_{\mathcal{O}_i/\mathcal{O}}$

qed

Proposition 9.5.10: If L/K is galois with Galois group Γ, then each $\hat{L}_i/\hat{K}$ is galois with Galois group $\Gamma_{\mathfrak{q}_i}$, and the respective inertia groups are equal: $I_{\mathfrak{q}_i} = I_{\mathfrak{n}_i}$.

decomposition group of $\mathfrak{q}_i$

Note 9.5.11: Passing to the completion is therefore similar to passing to the decomposition group.

transitive

Proof: Γ acts on $L \otimes_K \hat{K} \cong \prod_{i=1}^{r} \hat{L}_i$ $\Rightarrow$ permutes the factors $\Rightarrow$ on the $\mathfrak{n}_i$, compatibly with the action on the $\mathfrak{q}_i$.

$$A \supset \mathfrak{p} \qquad \mathcal{O}_i$$
$$B \supset \mathfrak{q}_i \longleftrightarrow \mathfrak{n}_i$$

$\Rightarrow \mathrm{Stab}_\Gamma(\mathfrak{n}_i) = \mathrm{Stab}_\Gamma(\mathfrak{q}_i) = \Gamma_{\mathfrak{q}_i}$.

$$|\Gamma_{\mathfrak{q}_i}| = \frac{|\Gamma|}{r} = \frac{[L/K]}{r} = [\hat{L}_i/\hat{K}].$$

$$(L \otimes_K \hat{K})^\Gamma = L^\Gamma \otimes_K \hat{K} = K \otimes_K \hat{K} \cong \hat{K}$$
$$\| $$
$$(\hat{L}_i)^{\Gamma_{\mathfrak{q}_i}}$$

$\Rightarrow$ $\hat{L}_i/\hat{K}$ galois with $\mathrm{Gal}(\hat{L}_i/\hat{K}) = \Gamma_{\mathfrak{q}_i}$.

Inertia group $I_{\mathfrak{q}_i} = \{\sigma \in \Gamma_{\mathfrak{q}_i} \mid \sigma \text{ induces the identity on } k(\mathfrak{q}_i)\}$ = inertia group of $\hat{L}_i/\hat{K}$.

$k(\mathfrak{q}_i) = k(\mathfrak{n}_i)$

qed.

Also: $K' := L^{I_{\mathfrak{q}_i}}$, $\mathfrak{q}_i' := \mathfrak{q}_i \cap K'$ $\Rightarrow$ the completion of K' w.r.t. $\mathfrak{q}_i'$ is $\hat{K}$.

9.6 Local and global fields

Recall that a Hausdorff topological space is called *locally compact* if every neighborhood of every point contains a compact neighborhood. For example $\mathbb{R}^n$ is locally compact, but an infinite dimensional Hilbert or Banach space is not. On locally compact spaces one can do analysis in much the same way as on $\mathbb{R}^n$.

Theorem 9.6.1: For any field K with an absolute value $|\ |$ the following are equivalent:

(a) K is locally compact.

(b) $|\ |$ is complete and, if it is nonarchimedean, the associated valuation is discrete and has finite residue field.

(c) K is isomorphic to a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for a prime p, and $|\ |$ is equivalent to the unique extension of the usual absolute value on that field.

Definition 9.6.2: Such a field K is called a *local field.*

Proof: (c) $\Rightarrow$ (a) $\underbrace{\mathbb{R}, \mathbb{Q}_p, \mathbb{F}_p((t))}_{K_0}$ are locally compact : $p^n\mathbb{Z}_p$ compact for any $n \in \mathbb{Z}$. $t^n\mathbb{F}_p[[t]]$

If K/K_0 is finite, then $K \cong K_0^n$ as top. space $\Rightarrow$ locally compact.

(a) $\Rightarrow$ (b) Let U be a compact nbhd of 0. Then $U \supset B_\varepsilon(0)$ for some $\varepsilon > 0$. $= \{x \in K : |x| \leq \varepsilon\}$.

(x_n) Cauchy sequence. Take n_0: $\forall n, m \geq n_0$: $d(x_n, x_m) \leq \varepsilon/2$.

$\Rightarrow \forall n \geq n_0$: $x_n \in B_\varepsilon(x_{n_0}) \subset U + x_{n_0}$ $\Rightarrow$ has an accumulation point $=$ limit.

So $|.|$ is complete.

Suppose $|\cdot|$ is nonarch. Pick $\pi \in K^\times : |\pi| < 1$. Let $\mathcal{O} = \{y \in K : |y| \leq 1\}$.

Let $\{x_i \mid i \in I\}$ be a system of rep's for $\mathcal{O}/\pi\mathcal{O}$.

Then $\pi\mathcal{O}$ is open and $\underline{\mathcal{O}} = \coprod_{i \in I} \underline{x_i + \pi\mathcal{O}}$ $\to$ $\underline{I \text{ is finite}}$

K locally compact, all $\pi^n \mathcal{O}$ for $n \geq 0$ fundamental system of closed neighborhoods

$\Rightarrow \exists n : \pi^n \mathcal{O}$ compact $\Rightarrow$ $\underline{\mathcal{O} \text{ compact}}$.

So $\mathcal{O}/\pi\mathcal{O}$ finite $\Rightarrow$ residue field is finite.

Also $\forall x \in K$ with $\underline{0 \leq v(x) < v(\pi)}$: $x \in \mathcal{O}$ and $\exists!\, i : x + \pi\mathcal{O} = x_i + \pi\mathcal{O}$

$\Rightarrow$ $\underline{v(x) = v(x_i)}$

$\Rightarrow v(K^\times) \cap [0, v(\pi)[$ finite

$\Rightarrow v(K^\times) \subset \mathbb{R}$ discrete.

(b) $\Rightarrow$ (c) If archimedean then $K \cong \mathbb{R}$ or $\mathbb{C}$ by ...

Take K nonarch. Case $\operatorname{char}(K) = 0$, that is $\mathbb{Q} \subset K$.

If $|\cdot|\big|_{\mathbb{Q}}$ is trivial, then $\mathbb{Q} \subset \mathcal{O}$ and $\mathbb{Q} \cap \mathfrak{m} = \{0\} \Rightarrow \mathbb{Q} \hookrightarrow \mathcal{O}/\mathfrak{m} = \text{residue field} \underset{\text{finite}}{\overset{\Downarrow}{}}$.

So $|\cdot|\big|_{\mathbb{Q}}$ is nontrivial $\Rightarrow$ equivalent to $|\cdot|_p$ for some p. WLOG: $|\cdot|\big|_{\mathbb{Q}} = |\cdot|_p$.

Then completeness $\mathbb{Q}_p \subset K$ and $p \in \mathfrak{m}$.

As above $\mathcal{O}/p\mathcal{O}$ finite, say $\mathcal{O} = \bigcup_{i \in I = \text{finite}} x_i + p\mathcal{O}$ $\Rightarrow$

Let $\Lambda := \prod_{i \in I} \mathbb{Z}_p \cdot x_i \subset \mathcal{O}$ $\Rightarrow \mathcal{O} = \Lambda + p\mathcal{O} = \Lambda + p\Lambda + p^2\mathcal{O} = \ldots = \Lambda + p^n\mathcal{O}$ for all n.

So Λ is dense in $\mathcal{O}$, compact $\Rightarrow$ closed $\Rightarrow \Lambda = \mathcal{O}$.

So $\mathcal{O}$ fin. gen. $\mathbb{Z}_p$-module $\Rightarrow K$ fin. ext. of $\mathbb{Q}_p$. ✓

Case $\operatorname{char}(K) = p > 0$. Take $t \in \mathfrak{m} \setminus \{0\}$,

repeat with $\mathbb{F}_p[[t]]$ in place of $\mathbb{Z}_p$.

qed

Remark 9.6.3: The characteristic of a nonarchimedean local field is either zero or equal to the characteristic of its residue field.

Definition 9.6.4: To exhibit the analogy we sometimes write $\mathbb{Q}_\infty := \mathbb{R}$ and denote the usual absolute value by $|\ |_\infty$.

Definition 9.6.5: A field that is isomorphic to a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for a prime p is called a *global field.*

Proposition 9.6.6: A field with an absolute value is a local field if and only if it is the completion of a global field at an absolute value.

Proof: "$\Longleftarrow$" K global field, $|\cdot|$ absolute value

$|\cdot|$ arch. $\Rightarrow \hat{K} = \mathbb{R}$ or $\mathbb{C}$ ✓

$|\cdot|$ non-arch. : K/K_0 finite $\Rightarrow$ Lemma: Then $|\cdot||_{K_0}$ is again nonarchimedean.

$K_0 = \mathbb{Q}$ or $\mathbb{F}_p(t)$

$K_0 = \mathbb{Q} \Rightarrow$ WLOG $|\cdot||_{\mathbb{Q}} = |\cdot|_p$. Then K = finite extension of $\mathbb{Q}_p$ by §9.5.

$K_0 = \mathbb{F}_p(t)$: Choose element $t \in K$ with $0 < |t| < 1$. Then t is transcendental over $\mathbb{F}_p$.

$\Rightarrow K_1 := \mathbb{F}_p(t) \subset K$ and $|\cdot||_{K_1}$ corresponds to $(t) \subset \mathbb{F}_p[[t]]$. . . .

Remark 9.6.7: There is a delicate interplay between properties of global field and properties of their associated local fields, which can go both ways.